

SEMIGROUP C*-ALGEBRAS AND AMENABILITY OF SEMIGROUPS

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ABSTRACT. We construct reduced and full semigroup C*-algebras for left cancellative semigroups. Our new construction covers particular cases already considered by A. Nica and also Toeplitz algebras attached to rings of integers in number fields due to J. Cuntz.

Moreover, we show how (left) amenability of semigroups can be expressed in terms of these semigroup C*-algebras in analogy to the group case.

1. INTRODUCTION

The construction of group C*-algebras provides examples of C*-algebras which are both interesting and challenging to study. If we restrict our discussion to discrete groups, then we could say that the idea behind the construction is to implement the algebraic structure of a given group in a concrete or abstract C*-algebra in terms of unitaries. It then turns out that the group and its group C*-algebra(s) are closely related in various ways, for instance with respect to representation theory or in the context of amenability.

Given the success and the importance of the construction of group C*-algebras, a very natural question is whether we can start with algebraic structures that are even more basic than groups, namely semigroups. And indeed, this question has been addressed by various authors. The start was made by L. Coburn who studied the C*-algebra of the additive semigroup of the natural numbers (see [Co1] and [Co2]). Then, just to mention some examples, a number of authors like L. Coburn, R. G. Douglas, R. Howe, D. G. Schaeffer and I. M. Singer studied C*-algebras of particular Toeplitz operators in [Co-Do], [C-D-S-S], [Dou] and [Do-Ho]. The original motivation came from index theory and related K-theoretic questions. Later on, G. Murphy further generalized this construction, first to positive cones in ordered abelian groups in [Mur1], then to arbitrary left cancellative semigroups in [Mur2] and [Mur3]. The basic idea behind the constructions mentioned so far is to replace unitary representations in the group case by isometric representations for left cancellative semigroups. However, it turns out that the full semigroup C*-algebras introduced by G. Murphy are very complicated and not suited for studying amenability. For instance, the full

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semigroup C^* -algebra of $\mathbb{N} \times \mathbb{N}$ in the sense of G. Murphy is not nuclear (see [Mur4], Theorem 6.2).

Apart from these constructions, A. Nica has introduced a different construction of semigroup C^* -algebras for positive cones in quasi-lattice ordered groups (see [Ni] and also [La-Rae]). His construction has the advantage that it leads to much more tractable C^* -algebras than the construction introduced by G. Murphy, so that A. Nica was able to study amenability questions using his new construction. The main difference between A. Nica's construction and the former ones is that A. Nica takes the right ideal structure of the semigroups into account in his construction, although in a rather implicit way.

Another source of inspiration is provided by so-called ring C^* -algebras (see [Cun], [Cu-Li1], [Cu-Li2] and [Li]). Namely, the author realized during his recent work [Li] that there are strong parallels between the construction of ring C^* -algebras and semigroup C^* -algebras. The restriction A. Nica puts on his semigroups by only considering positive cones in quasi-lattice ordered groups would correspond in the ring case to considering rings for which every ideal is principal. This observation indicates that the ideal structure (of the ring or semigroup) should play an important role in more general constructions. This idea has been worked out in the case of rings in [Li]. Moreover, it was explained in Appendix A.2 of [Li] how the analogous idea leads to a generalization of A. Nica's construction to arbitrary left cancellative semigroups.

Independently from this construction of semigroup C^* -algebras, J. Cuntz has modified the construction of ring C^* -algebras from [Cu-Li1] and [Cu-Li2] and has introduced so-called Toeplitz algebras for certain rings from algebraic number theory (rings of integers in number fields). The motivation was to improve the functorial properties of ring C^* -algebras. And again, the crucial idea behind the construction is to make use of the ideal structure of the rings of interest. This first step was due to J. Cuntz (before the work [C-D-L]), and he presented these ideas and the results on functoriality in a talk at the "Workshop on C^* -algebras" in Nottingham which took place in September 2010.

As a next step, J. Cuntz, C. Deninger and M. Laca study these Toeplitz algebras in [C-D-L] and they show that the Toeplitz algebra of the ring of integers in a number field can be identified via a canonical representation with the reduced semigroup C^* -algebra of the $ax + b$ -semigroup over the ring. This indicates that there is a strong connection between these Toeplitz algebras and semigroup C^* -algebras.

And indeed, it turns out that if we apply the construction of full semigroup C^* -algebras in [Li] to the $ax + b$ -semigroups over rings of integers, then we arrive at universal C^* -algebras which are canonically isomorphic to these Toeplitz algebras. As pointed out in [C-D-L], the most interesting examples arise from rings which do not have the property that every ideal is principal (i.e. the class number of the number field is strictly bigger than 1). For these rings or rather the corresponding

$ax + b$ -semigroups, it is not possible to apply A. Nica's construction. This explains the need for a generalization of A. Nica's work.

So, to summarize, the motivation behind our construction of semigroup C*-algebras is twofold: On the one hand, we would like to provide a general framework for A. Nica's constructions as well as the Toeplitz algebras due to J. Cuntz so that these constructions can be naturally thought of as particular cases of our general construction (this is explained in § 2). On the other hand, we would like to obtain constructions which are more tractable than those of G. Murphy and which allow us to characterize amenability of semigroups very much in the same spirit as in the group case (see § 4). To establish this connection with amenability, we first have to modify our construction of full semigroup C*-algebras in the case of subsemigroups of groups (see § 3).

Of course, there are not only C*-algebras associated with groups, but also C*-algebras attached to dynamical systems. So another question would be whether we can also construct C*-algebras for semigroup actions by automorphisms. We only touch upon this question in § 2.2.

I would like to thank J. Cuntz for interesting and helpful discussions and for providing access to the preprint [C-D-L] due to him, C. Deninger and M. Laca. I also thank M. Norling who has pointed me towards a missing relation in the definition of full semigroup C*-algebras for subsemigroups of groups. This has led me to the modified construction introduced in § 3.

2. CONSTRUCTIONS

2.1. Semigroup C*-algebras. By a semigroup, we mean a set P equipped with a binary operation $P \times P \rightarrow P; (p, q) \mapsto pq$ which is associative, i.e. $(p_1 p_2) p_3 = p_1 (p_2 p_3)$. We always assume that our semigroup has a unit element, i.e. there exists $e \in P$ such that $ep = pe = p$ for all $p \in P$. All semigroup homomorphisms shall preserve unit elements. We only consider discrete semigroups. A semigroup P is called left cancellative if for every p, x and y in P , $px = py$ implies $x = y$.

As mentioned in the introduction, the basic idea behind the construction of semigroup C*-algebras is to represent semigroup elements by isometries. This means that if we let Isom be the semigroup of the necessarily unital semigroup C*-algebra associated with the semigroup P , then we would like to have a semigroup homomorphism $P \rightarrow \text{Isom}$. This requirement explains why we restrict our discussion to left cancellative semigroups: Since Isom is always a left cancellative semigroup, this homomorphism $P \rightarrow \text{Isom}$ can only be faithful if P itself is left cancellative.

Given a left cancellative semigroup P , we can construct its left regular representation as follows:

Let $\ell^2(P)$ be the Hilbert space of square summable complex-valued functions on P . $\ell^2(P)$ comes with the canonical orthonormal basis $\{\varepsilon_x: x \in P\}$ given by $\varepsilon_x(y) = \delta_{x,y}$ where $\delta_{x,y}$ is 1 if $x = y$ and 0 if $x \neq y$. Let us define for every $p \in P$ an isometry V_p by setting $V_p \varepsilon_x = \varepsilon_{px}$. Here we have made use of our assumption that our semigroup P is left cancellative. It ensures that the assignment $\varepsilon_x \mapsto \varepsilon_{px}$ indeed extends to an isometry. Now the reduced semigroup C*-algebra of P is simply given as the sub-C*-algebra of $\mathcal{L}(\ell^2(P))$ generated by these isometries $\{V_p: p \in P\}$. We denote this concrete C*-algebra by $C_r^*(P)$, i.e. we set

Definition 2.1. $C_r^*(P) := C^*(\{V_p: p \in P\}) \subseteq \mathcal{L}(\ell^2(P))$.

So $C_r^*(P)$ is really a very natural object: It is the C*-algebra generated by the left regular representation of P . This C*-algebra $C_r^*(P)$ is called the reduced semigroup C*-algebra of P in analogy to the group case. But we remark that this C*-algebra is also called the Toeplitz algebra of P by various authors.

We now turn to the construction of full semigroup C*-algebras. As explained in the introduction, we will make use of right ideals of our semigroups to construct full semigroup C*-algebras. So we first have to choose a family of right ideals.

Given a semigroup P , every semigroup element $p \in P$ gives rise to the map $P \rightarrow P; x \mapsto px$. It is simply given by left multiplication with p . Given a subset X of P and an element $p \in P$, we set

$$(1) \quad pX := \{px: x \in X\} \text{ and } p^{-1}X := \{y \in P: py \in X\}.$$

In other words, pX is the image and $p^{-1}X$ is the pre-image of X under left multiplication with p . A subset X of P is called a right ideal if it is closed under right multiplication with arbitrary semigroup elements, i.e. if for every $x \in X$ and $p \in P$, the product xp always lies in X .

The semigroup P is left cancellative if and only if for every $p \in P$, left multiplication with p defines an injective map. For the rest of this section, let P always be a left cancellative semigroup.

Let \mathcal{J} be the smallest family of right ideals of P containing P and \emptyset , i.e.

$$(2) \quad P \in \mathcal{J}, \emptyset \in \mathcal{J},$$

and closed under left multiplication, taking pre-images under left multiplication,

$$(3) \quad X \in \mathcal{J}, p \in P \Rightarrow pX, p^{-1}X \in \mathcal{J},$$

as well as finite intersections,

$$(4) \quad X, Y \in \mathcal{J} \Rightarrow X \cap Y \in \mathcal{J}.$$

It is not difficult to find out how right ideals in \mathcal{J} typically look like. Actually, it follows directly from the definitions that

$$(5) \quad \mathcal{J} = \left\{ \bigcap_{j=1}^N (q_{j,1})^{-1} p_{j,1} \cdots (q_{j,n_j})^{-1} p_{j,n_j} P: N, n_j \in \mathbb{Z}_{>0}; p_{j,k}, q_{j,k} \in P \right\} \cup \{\emptyset\}.$$

The elements in \mathcal{J} are called constructible right ideals. If we want to keep track of the semigroup, we write \mathcal{J}_P for the family of constructible right ideals of the semigroup P . We will see in (32) that it is not necessary to ask for (4).

With the help of this family of right ideals, we can now construct the full semigroup C*-algebra of P . The idea is to ask for a projection-valued spectral measure, defined for elements in the family \mathcal{J} and taking values in projections in our C*-algebra.

Definition 2.2. *The full semigroup C*-algebra of P is the universal C*-algebra generated by isometries $\{v_p: p \in P\}$ and projections $\{e_X: X \in \mathcal{J}\}$ satisfying the following relations:*

$$\begin{aligned} I.(i) \ v_{pq} &= v_p v_q & I.(ii) \ v_p e_X v_p^* &= e_{pX} \\ II.(i) \ e_P &= 1 & II.(ii) \ e_\emptyset &= 0 & II.(iii) \ e_{X \cap Y} &= e_X \cdot e_Y \end{aligned}$$

for all p, q in P and X, Y in \mathcal{J} .

We denote this universal C*-algebra by $C^*(P)$, i.e.

$$C^*(P) := C^* \left(\left\{ v_p: p \in P \right\} \cup \left\{ e_X: X \in \mathcal{J} \right\} \left| \begin{array}{l} v_p \text{ are isometries} \\ \text{and } e_X \text{ are projections} \\ \text{satisfying I and II.} \end{array} \right. \right)$$

One remark about notation: For the sake of readability, we sometimes write $e_{[X]}$ for e_X in case the expression in the index gets very long.

Of course, the question is: Where do all these relations come from? The idea is that we can think of $C^*(P)$ as a universal model of the reduced semigroup C*-algebra $C_r^*(P)$. To make this precise, let us again consider concrete operators on $\ell^2(P)$. We have already defined the isometries V_p for $p \in P$. For every subset X of P , let E_X be the orthogonal projection onto $\ell^2(X) \subseteq \ell^2(P)$. In other words, let $\mathbb{1}_X$ be the characteristic function of X defined on P , i.e. $\mathbb{1}_X(p) = 1$ if $p \in X$ and $\mathbb{1}_X(p) = 0$ if $p \notin X$. Then $\mathbb{1}_X$ is an element of $\ell^\infty(P)$ which is mapped to E_X under the canonical representation of $\ell^\infty(P)$ as multiplication operators on $\ell^2(P)$. As with the projections e_X , we will sometimes write $E_{[X]}$ for E_X if the subscript becomes very long. It is now easy to check that the two families $\{V_p: p \in P\}$ and $\{E_X: X \in \mathcal{J}\}$ satisfy relations I and II (with V_p in place of v_p and E_X in place of e_X). This explains the origin of these relations. At the same time, we obtain by universal property of $C^*(P)$ a non-zero homomorphism $\lambda: C^*(P) \rightarrow \mathcal{L}(\ell^2(P))$ sending v_p to V_p and e_X to E_X for every $p \in P$ and $X \in \mathcal{J}$. This homomorphism is called the left regular representation of $C^*(P)$. In particular, we see that $C^*(P)$ is not the zero C*-algebra. We will see later on (compare (11)) that the image of λ is actually the reduced semigroup C*-algebra $C_r^*(P)$.

Remark 2.3. Actually, the requirement that \mathcal{J} should be closed under taking pre-images under left multiplications is not needed in the construction, and it does not appear in the first version of semigroup C*-algebras in [Li], Appendix A.2. The original reason why we added this extra requirement is that we wanted our construction of full semigroup C*-algebras to include the construction of Toeplitz algebras for rings of integers in number fields by J. Cuntz. However, for such semigroups, it

is not necessary to consider pre-images in the following sense: Let \mathcal{J}' denote the family of right ideals defined in the same way as \mathcal{J} but without the property that \mathcal{J}' is closed under pre-images under left multiplication. For the $ax + b$ -semigroups over rings of integers, it turns out that it does not matter whether we take \mathcal{J} or \mathcal{J}' in Definition 2.2 because the resulting C^* -algebras are canonically isomorphic. But for general semigroups, it is more convenient to work with \mathcal{J} as we will see.

Let us also discuss a useful modification of these full semigroup C^* -algebras. We first reformulate relation II.(iii): We have canonical lattice structures on the set of right ideals of P (let $X \wedge Y = X \cap Y$ and $X \vee Y = X \cup Y$ for right ideals X and Y) and on the set of commuting projections in a C^* -algebra (let $e \wedge f = ef$ and $e \vee f = e + f - e \wedge f$ for commuting projections e and f). So relation II.(iii) simply tells us that the projections $\{e_X : X \in \mathcal{J}\}$ commute and that the assignment $\mathcal{J} \ni X \mapsto e_X \in \text{Proj}(C^*(P))$ is \wedge -compatible. Given this interpretation, an obvious question is whether we can modify our construction so that the analogous assignment becomes \vee -compatible as well. This is indeed possible. The first step is to enlarge the family \mathcal{J} so that it is closed under finite unions as well. Let $\mathcal{J}^{(\cup)}$ be the smallest family of right ideals of P satisfying the conditions (2) – (4) and the extra condition

$$(6) \quad X, Y \in \mathcal{J}^{(\cup)} \Rightarrow X \cup Y \in \mathcal{J}^{(\cup)}.$$

Again, it follows from our definition that

$$(7) \quad \mathcal{J}^{(\cup)} = \left\{ \bigcup_{i=1}^M \bigcap_{j=1}^N (q_{j,1}^{(i)})^{-1} p_{j,1}^{(i)} \cdots (q_{j,n_j}^{(i)})^{-1} p_{j,n_j}^{(i)} P : M, N, n_j \in \mathbb{Z}_{>0}; p_{j,k}^{(i)}, q_{j,k}^{(i)} \in P \right\} \cup \{\emptyset\}.$$

We can now modify Definition 2.2 by replacing \mathcal{J} by $\mathcal{J}^{(\cup)}$ and adding to the relations the extra relation $e_{X \cup Y} = e_X + e_Y - e_{X \cap Y}$ for all $X, Y \in \mathcal{J}^{(\cup)}$. The corresponding universal C^* -algebra is then denoted by $C^{*(\cup)}(P)$.

Definition 2.4.

$$C^{*(\cup)}(P) := C^* \left(\left\{ v_p : p \in P \right\} \cup \left\{ e_X : X \in \mathcal{J}^{(\cup)} \right\} \right) \left| \begin{array}{l} v_p \text{ are isometries} \\ \text{and } e_X \text{ are projections} \\ \text{satisfying I and II}^{(\cup)} \end{array} \right.$$

with the relations

$$\begin{aligned} I.(i) \quad v_{pq} &= v_p v_q & I.(ii) \quad v_p e_X v_p^* &= e_{pX} \\ II^{(\cup)}.(i) \quad e_P &= 1 & II^{(\cup)}.(ii) \quad e_\emptyset &= 0 \\ II^{(\cup)}.(iii) \quad e_{X \cap Y} &= e_X \cdot e_Y & II^{(\cup)}.(iv) \quad e_{X \cup Y} &= e_X + e_Y - e_{X \cap Y}. \end{aligned}$$

It is immediate from our definitions that $C^{*(\cup)}(P)$ is a quotient of $C^*(P)$, or in other words, that we always have a canonical homomorphism $\pi^{(\cup)} : C^*(P) \rightarrow C^{*(\cup)}(P)$ sending $C^*(P) \ni v_p$ to $v_p \in C^{*(\cup)}(P)$ and $C^*(P) \ni e_X$ to $e_X \in C^{*(\cup)}(P)$ for all $p \in P$ and $X \in \mathcal{J} \subseteq \mathcal{J}^{(\cup)}$. Relation II^(u).(iv) implies that $\pi^{(\cup)}$ is always surjective.

As for the relations defining $C^*(P)$, it is immediate that the relations I and II^(U) (with V_p in place of v_p and E_X in place of e_X) are satisfied by the concrete operators $\{V_p : p \in P\}$ and $\{E_X : X \in \mathcal{J}^{(U)}\}$ on $\ell^2(P)$ (E_X is the orthogonal projection onto $\ell^2(X) \subseteq \ell^2(P)$ as above). So we again obtain by universal property of $C^{*(U)}(P)$ a non-zero homomorphism $\lambda^{(U)} : C^{*(U)}(P) \rightarrow \mathcal{L}(\ell^2(P))$ sending v_p to V_p and e_X to E_X for every $p \in P$ and $X \in \mathcal{J}^{(U)}$. This again implies that $C^{*(U)}(P)$ is not the zero C*-algebra. Moreover, we obtain by construction a commutative diagram

$$(8) \quad \begin{array}{ccc} C^*(P) & & \\ \pi^{(U)} \downarrow & \searrow \lambda & \\ C^{*(U)}(P) & \xrightarrow{\lambda^{(U)}} & \mathcal{L}(\ell^2(P)) \end{array}$$

2.2. Semigroup crossed products by automorphisms. At this point, we also introduce semigroup crossed products by automorphisms. Let P be a left cancellative semigroup and D a unital C*-algebra. Moreover, let $\alpha : P \rightarrow \text{Aut}(A)$ be a semigroup homomorphism.

We then define the full semigroup crossed product of A by P with respect to α as the (up to isomorphism unique) unital C*-algebra $A \rtimes_\alpha^a P$ which comes with two unital homomorphisms $\iota_A : A \rightarrow A \rtimes_\alpha^a P$ and $\iota_P : C^*(P) \rightarrow A \rtimes_\alpha^a P$ satisfying

$$\iota_A(\alpha_p(a))\iota_P(v_p) = \iota_P(v_p)\iota_A(a) \text{ for all } a \in A, p \in P$$

such that the following universal property is fulfilled:

Whenever T is a unital C*-algebra and $\varphi_A : A \rightarrow T$, $\varphi_P : C^*(P) \rightarrow T$ are unital homomorphisms satisfying the covariance relation

$$(9) \quad \varphi_A(\alpha_p(a))\varphi_P(v_p) = \varphi_P(v_p)\varphi_A(a) \text{ for all } a \in A, p \in P,$$

there is a unique homomorphism $\varphi_A \rtimes \varphi_P : A \rtimes_\alpha^a P \rightarrow T$ with

$$(\varphi_A \rtimes \varphi_P) \circ \iota_A = \varphi_A \text{ and } (\varphi_A \rtimes \varphi_P) \circ \iota_P = \varphi_P.$$

We could also use $C^{*(U)}(P)$ instead of $C^*(P)$ in the construction of the semigroup crossed product by automorphisms, and the result would be another C*-algebra, say $A \rtimes_{\alpha}^{a,(U)} P$, with the corresponding universal property. We will see in Lemma 2.15 that these universal C*-algebras really exist. By construction, we have a canonical homomorphism $\pi_{(A,P,\alpha)}^{(U)} : A \rtimes_\alpha^a P \rightarrow A \rtimes_{\alpha}^{a,(U)} P$. This homomorphism is surjective as the canonical homomorphism $\pi^{(U)} : C^*(P) \rightarrow C^{*(U)}(P)$ is surjective. Of course, if $\text{tr} : P \rightarrow \text{Aut}(\mathbb{C})$ denotes the trivial action, then

$$C^*(P) \cong \mathbb{C} \rtimes_{\text{tr}} P, \quad C^{*(U)}(P) \cong \mathbb{C} \rtimes_{\text{tr}}^{(U)} P,$$

and under these canonical identifications, $\pi_{(C,P,\text{tr})}^{(U)}$ becomes the canonical homomorphism $\pi^{(U)} : C^*(P) \rightarrow C^{*(U)}(P)$.

We remark that there is a different notion of semigroup crossed products by endomorphisms which is for instance explained in [La], [La-Rae], § 2 or in [Li], Appendix A.1.

We denote semigroup crossed products by endomorphisms by \rtimes^e to distinguish them from our construction. We will see that there is a close relationship between these two sorts of semigroup crossed products.

G. Murphy has already introduced semigroup crossed products by automorphisms in [Mur2] and [Mur3]. However, as in the case of semigroup C*-algebras, G. Murphy's construction leads to very complicated C*-algebras which are not tractable even in very simple cases. But G. Murphy has also constructed concrete representations, and these can be used to define reduced semigroup crossed products by automorphisms: Take a faithful representation of D on a Hilbert space H , say $i : A \rightarrow \mathcal{L}(H)$. Form the tensor product $H \otimes \ell^2(P)$. Then define for every a in A a bounded operator by the formula $\eta \otimes \varepsilon_x \mapsto i(\alpha_x^{-1}(a))(\eta) \otimes \varepsilon_x$ for every $\eta \in H$ and $x \in P$. It is straightforward to check that these operators give rise to a homomorphism $i_A : A \rightarrow \mathcal{L}(H \otimes \ell^2(P))$ and that i_A and $i_P := \text{id}_H \otimes \lambda : C^*(P) \rightarrow \mathcal{L}(H \otimes \ell^2(P))$ satisfy the covariance relation (9). Thus we obtain by universal property of $A \rtimes_\alpha^a P$ a homomorphism $\lambda_{(A,P,\alpha)} := i_A \rtimes i_P : A \rtimes_\alpha^a P \rightarrow \mathcal{L}(H \otimes \ell^2(P))$. We set $A \rtimes_{\alpha,r}^a P := \lambda_{(A,P,\alpha)}(A \rtimes_\alpha^a P)$ and call this algebra the reduced semigroup crossed product of A by P with respect to α . Using the same faithful representation i of A , the induced homomorphism $i_A : A \rightarrow \mathcal{L}(H \otimes \ell^2(P))$ and the homomorphism $\text{id}_H \otimes \lambda^{(\cup)} : C^{*(\cup)}(P) \rightarrow \mathcal{L}(H \otimes \ell^2(P))$, we can also construct a homomorphism $\lambda_{(A,P,\alpha)}^{(\cup)} : A \rtimes_\alpha^{a,(\cup)} P \rightarrow \mathcal{L}(H \otimes \ell^2(P))$. Again, by universal property of $A \rtimes_\alpha^a P$, $\lambda_{(A,P,\alpha)} = \lambda_{(A,P,\alpha)}^{(\cup)} \circ \pi_{(A,P,\alpha)}^{(\cup)}$, so there is no difference between $A \rtimes_{\alpha,r}^{a,(\cup)} P := \lambda_{(A,P,\alpha)}^{(\cup)}(A \rtimes_\alpha^{a,(\cup)} P)$ and $A \rtimes_{\alpha,r}^a P$.

Remark 2.5. Of course, we can consider right cancellative semigroups instead of left cancellative ones. Replacing left multiplication by right multiplication and right ideals by left ideals, we obtain analogous constructions. Alternatively, given a right cancellative semigroup P , we can go over to the opposite semigroup P^{op} consisting of the same underlying set P equipped with a new binary operation \bullet given by $p \bullet q := qp$. It is immediate that P^{op} is left cancellative and our constructions apply.

With the obvious modifications, our analysis of C*-algebras associated with left cancellative semigroups (which is going to come) carries over to right cancellative semigroups.

2.3. Direct consequences of the definitions. First of all, each of the C*-algebras $C^*(P)$ and $C^{*(\cup)}(P)$ contains a distinguished sub-C*-algebra, namely the one generated by the projections $\{e_X : X \in \mathcal{J}\}$ or $\{e_X : X \in \mathcal{J}^{(\cup)}\}$. Let us denote these sub-C*-algebras by $D(P)$ and $D^{(\cup)}(P)$, i.e.

$$\begin{aligned} D(P) &:= C^*(\{e_X : X \in \mathcal{J}\}) \subseteq C^*(P) \\ D^{(\cup)}(P) &:= C^*(\{e_X : X \in \mathcal{J}^{(\cup)}\}) \subseteq C^{*(\cup)}(P). \end{aligned}$$

We first observe that

$$(10) \quad \pi^{(\cup)}(D(P)) = D^{(\cup)}(P).$$

The inclusion “ \subseteq ” is clear as $\mathcal{J} \subseteq \mathcal{J}^{(\cup)}$, and the reverse inclusion “ \supseteq ” follows immediately from relation $\text{II}^{(\cup)}.$ (iv) and the concrete description of $\mathcal{J}^{(\cup)}$ in (7).

Moreover, we have the following

Lemma 2.6. *The families $\{e_X: X \in \mathcal{J}\}$ and $\{e_X: X \in \mathcal{J}^{(\cup)}\}$ consist of commuting projections and are multiplicatively closed.*

Proof. This follows immediately from relation II.(iii) and II.^(\cup).(iii), respectively. \square

Corollary 2.7. *$D(P)$ and $D^{(\cup)}(P)$ are commutative C*-algebras.*

Moreover, $D(P) = \overline{\text{span}}(\{e_X: X \in \mathcal{J}\})$ and $D^{(\cup)}(P) = \overline{\text{span}}(\{e_X: X \in \mathcal{J}^{(\cup)}\})$.

Furthermore, as another consequence of the definitions, we derive

Lemma 2.8. *For every $p \in P$ and $X \in \mathcal{J}$ ($X \in \mathcal{J}^{(\cup)}$), we have $v_p^* e_X v_p = e_{p^{-1}X}$ in $C^*(P)$ ($C^{*(\cup)}(P)$).*

Proof. The proof is the same for $C^*(P)$ and $C^{*(\cup)}(P)$. Take $p \in P$ and $X \in \mathcal{J}$ ($X \in \mathcal{J}^{(\cup)}$). We then have $v_p^* e_X v_p = v_p^* e_X v_p v_p^* v_p = v_p^* e_X e_p v_p = v_p^* e_{X \cap pP} v_p = v_p^* e_{p(p^{-1}X)} v_p = v_p^* v_p e_{p^{-1}X} v_p^* v_p = e_{p^{-1}X}$. \square

Corollary 2.9. *For every $p \in P$, conjugation by $v_p^* \in C^*(P)$ ($v_p^* \in C^{*(\cup)}(P)$) induces a homomorphism on $D(P)$ ($D^{(\cup)}(P)$).*

Proof. This is a direct consequence of the previous lemma. \square

From Lemma 2.8 and the description of \mathcal{J} given in (5), we immediately deduce

Corollary 2.10. *$C^*(P)$ is generated as a C*-algebra by the isometries $\{v_p: p \in P\}$.*

We also obtain the analogous statement for $C^{*(\cup)}(P)$:

Corollary 2.11. *$C^{*(\cup)}(P)$ is generated as a C*-algebra by $\{v_p: p \in P\}$.*

Proof. This either follows analogously from Lemma 2.8 for $C^{*(\cup)}(P)$ and the explicit description of $\mathcal{J}^{(\cup)}$ in (7) or with the help of the last corollary and the surjection $\pi^{(\cup)}: C^*(P) \rightarrow C^{*(\cup)}(P)$. \square

Now, it follows from Corollary 2.10 that the image of the left regular representation $\lambda: C^*(P) \rightarrow \mathcal{L}(\ell^2(P))$ is precisely the reduced semigroup C*-algebra $C_r^*(P)$. This means that we can rewrite the commutative triangle (8) more accurately as follows:

$$(11) \quad \begin{array}{ccc} C^*(P) & & \\ \pi^{(\cup)} \downarrow & \searrow \lambda & \\ C^{*(\cup)}(P) & \xrightarrow{\lambda^{(\cup)}} & C_r^*(P) \end{array}$$

As we did before for the full semigroup C^* -algebras, we consider a canonical sub- C^* -algebra of $C_r^*(P)$:

Definition 2.12. $D_r(P) := C^*(\{E_X : X \in \mathcal{J}\}) \subseteq \mathcal{L}(\ell^2(P))$.

Recall that E_X is the orthogonal projection onto the subspace $\ell^2(X) \subseteq \ell^2(P)$.

It is immediately clear that $\lambda(D(P)) = D_r(P)$, so that $D_r(P)$ is a sub- C^* -algebra of $C_r^*(P)$. $D_r(P)$ is obviously commutative and we have $D_r(P) = \overline{\text{span}}(\{E_X : X \in \mathcal{J}\})$ since $\{E_X : X \in \mathcal{J}\}$ is multiplicatively closed. Because of $\lambda(D(P)) = D_r(P)$, the commutative triangle (11), restricted to the distinguished commutative sub- C^* -algebras, yields the commutative triangle

$$(12) \quad \begin{array}{ccc} D(P) & & \\ \pi^{(\cup)} \downarrow & \searrow \lambda & \\ D^{(\cup)}(P) & \xrightarrow{\lambda^{(\cup)}} & D_r(P) \end{array}$$

Another direct consequence of our constructions is that we can alternatively describe our constructions as semigroup crossed products by endomorphisms. For the reader's convenience, we recall the notion of semigroup crossed products by endomorphisms. Let P be a discrete semigroup and D a unital C^* -algebra. Further assume that $\tau : P \rightarrow \text{End}(D)$ is a semigroup homomorphism from P to the semigroup $\text{End}(D)$ of (not necessarily unital) endomorphisms of D .

Definition 2.13. *The semigroup crossed product $D \rtimes_\tau^e P$ is the up to canonical isomorphism unique unital C^* -algebra which comes with a unital homomorphism $i_D : D \rightarrow D \rtimes_\tau^e P$ and a semigroup homomorphism $i_P : P \rightarrow \text{Isom}(D \rtimes_\tau^e P)$ subject to the condition $i_P(p)i_D(d)i_P(p)^* = i_D(\tau_p(d))$ for all $p \in P, d \in D$ and satisfying the following universal property:*

Whenever T is a unital C^ -algebra, $j_D : D \rightarrow T$ is a unital homomorphism and $j_P : P \rightarrow \text{Isom}(T)$ is a semigroup homomorphism such that the covariance relation*

$$(13) \quad j_P(p)j_D(d)j_P(p)^* = j_D(\tau_p(d)) \text{ for all } p \in P, d \in D$$

is fulfilled, there is a unique homomorphism $j_D \rtimes j_P : D \rtimes_\tau^e P \rightarrow T$ with $(j_D \rtimes j_P) \circ i_D = j_D$ and $(j_D \rtimes j_P) \circ i_P = j_P$. Here $\text{Isom}(D \rtimes_\tau^e P)$ and $\text{Isom}(T)$ are the semigroups of isometries in $D \rtimes_\tau^e P$ and T , respectively.

Existence of $D \rtimes_\tau^e P$ is shown in [La-Rae], § 2; their condition (iii) is equivalent to uniqueness of $j_D \rtimes j_P$.

Now, in our situation, there are canonical actions (i.e. semigroup homomorphisms) $\tau : P \rightarrow \text{End}(D(P))$ and $\tau^{(\cup)} : P \rightarrow \text{End}(D^{(\cup)}(P))$ given by $P \ni p \mapsto v_p \sqcup v_p^*$. Conjugation by v_p gives rise to a homomorphism of $C^*(P)$ because v_p is an isometry, and $D(P)$ ($D^{(\cup)}(P)$) is invariant under these homomorphisms by relation I.(ii). When we form the corresponding semigroup crossed products by endomorphisms, we obtain

Lemma 2.14. $C^*(P)$ is canonically isomorphic to $D(P) \rtimes_\tau^e P$, and $C^{*(\cup)}(P)$ is canonically isomorphic to $D^{(\cup)}(P) \rtimes_{\tau^{(\cup)}}^e P$.

Proof. Using the universal property of $C^*(P)$ and $D(P) \rtimes_\tau^e P$, we can construct mutually inverse homomorphisms $C^*(P) \rightleftharpoons D(P) \rtimes_\tau^e P$. It is clear that the isometries $\{i_P(p): p \in P\} \subseteq D(P) \rtimes_\tau^e P$ and the projections $\{i_{D(P)}(e_X): X \in \mathcal{J}\} \subseteq D(P) \rtimes_\tau^e P$ satisfy relations I and II (in place of the v_p s and e_X s), so that there exists a homomorphism $C^*(P) \rightarrow D(P) \rtimes_\tau^e P$ sending v_p to $i_P(p)$ and e_X to $i_{D(P)}(e_X)$ for all $p \in P$ and $X \in \mathcal{J}$. Conversely, $C^*(P)$ together with the inclusion $D(P) \hookrightarrow C^*(P)$ and the semigroup homomorphism $P \ni p \mapsto v_p \in \text{Isom}(C^*(P))$ satisfies the covariance relation (13) because of relation I.(ii). Hence there exists a homomorphism $D(P) \rtimes_\tau^e P \rightarrow C^*(P)$ sending $i_P(p)$ to v_p and $i_{D(P)}(e_X)$ to e_X for all $p \in P$ and $X \in \mathcal{J}$. By construction, these two homomorphisms are inverse to one another.

Similarly, a comparison of the universal properties yields a canonical identification $C^{*(\cup)}(P) \cong D^{(\cup)}(P) \rtimes_{\tau^{(\cup)}}^e P$. \square

More generally, we can also describe $D \rtimes_\alpha^a P$ and $D \rtimes_{\alpha^{a,(\cup)}}^{a,(\cup)} P$ as crossed products.

Lemma 2.15. $A \rtimes_\alpha^a P$ and $A \rtimes_{\alpha^{a,(\cup)}}^{a,(\cup)} P$ exist and are canonically isomorphic to $(A \otimes D(P)) \rtimes_{\alpha \otimes \tau}^e P$ and $(A \otimes D^{(\cup)}(P)) \rtimes_{\alpha \otimes \tau^{(\cup)}}^e P$, respectively.

Proof. By construction, $A \rtimes_\alpha^a P$ and $(A \otimes D(P)) \rtimes_{\alpha \otimes \tau}^e P$ have the same universal property. (Note that relation (9) implies that $\iota_A(A)$ and $\iota_P(D(P))$ in $A \rtimes_\alpha^a P$ commute.) As $(A \otimes D(P)) \rtimes_{\alpha \otimes \tau}^e P$ exists by [La-Rae], Proposition 2.1, we have proven our assertions about $A \rtimes_\alpha^a P$. An analogous argument applies to $A \rtimes_{\alpha^{a,(\cup)}}^{a,(\cup)} P$. \square

Another observation is that our constructions behave nicely with respect to direct products of semigroups.

Lemma 2.16. Given two left cancellative semigroups P and Q , there are canonical isomorphisms

$$\begin{aligned} C^*(P \times Q) &\cong C^*(P) \otimes_{\max} C^*(Q) \text{ given by } v_{(p,q)} \mapsto v_p \otimes v_q \\ \text{and } C_r^*(P \times Q) &\cong C_r^*(P) \otimes_{\min} C_r^*(Q) \text{ given by } V_{(p,q)} \mapsto V_p \otimes V_q. \end{aligned}$$

Proof. For the first identification, we just have to compare the universal properties of these C*-algebras. The second identification is given by conjugation by the unitary $\ell^2(P) \otimes \ell^2(Q) \rightarrow \ell^2(P \times Q); \varepsilon_x \otimes \varepsilon_y \mapsto \varepsilon_{(x,y)}$. \square

Remark 2.17. We can also identify $C^{*(\cup)}(P \times Q)$ with $C^{*(\cup)}(P) \otimes_{\max} C^{*(\cup)}(Q)$ via $v_{(p,q)} \mapsto v_p \otimes v_q$. The problem is to show that there is a homomorphism $D^{(\cup)}(P \times Q) \rightarrow C^{*(\cup)}(P) \otimes_{\max} C^{*(\cup)}(Q)$ which sends for all $X \in \mathcal{J}_P$ and $Y \in \mathcal{J}_Q$ the projection $e_{X \times Y}$ to $e_X \otimes e_Y$. This has to be the case as we want that $v_{(p,q)}$ is sent to $v_p \otimes v_q$ for every $p \in P$ and $q \in Q$. Once we know that such a homomorphism $D^{(\cup)}(P \times Q) \rightarrow C^{*(\cup)}(P) \otimes_{\max} C^{*(\cup)}(Q)$ exists, the result follows from the universal property of $C^{*(\cup)}(P \times Q)$.

$Q) \rightarrow C^{*(\cup)}(P) \otimes_{\max} C^{*(\cup)}(Q)$ exists, we can easily construct, using Lemma 2.14, the desired homomorphism $C^{*(\cup)}(P \times Q) \rightarrow C^{*(\cup)}(P) \otimes_{\max} C^{*(\cup)}(Q)$ satisfying $v_{(p,q)} \mapsto v_p \otimes v_q$. It is also easy to construct the inverse homomorphism $C^{*(\cup)}(P \times Q) \leftarrow C^{*(\cup)}(P) \otimes_{\max} C^{*(\cup)}(Q)$. It turns out that such a desired homomorphism $D^{(\cup)}(P \times Q) \rightarrow C^{*(\cup)}(P) \otimes_{\max} C^{*(\cup)}(Q)$ indeed exists (see Corollary 2.23). But the proof will have to wait until we have studied in more detail the relationship between $D^{(\cup)}(P)$ and $D_r(P)$.

2.4. Examples. Of course, if P happens to be a group, then our constructions coincide with the usual constructions of group C^* -algebras or ordinary crossed products. To be more precise, if P is a group, then the canonical homomorphism $\pi^{(\cup)} : C^*(P) \rightarrow C^{*(\cup)}(P)$ is an isomorphism. Moreover, $C^*(P)$ and $C_r^*(P)$ can be canonically identified with the full and the reduced group C^* -algebra of the group P . Analogously, for every unital C^* -algebra A and every (semi)group homomorphism $P \rightarrow \text{Aut}(A)$, the canonical homomorphism $\pi_{(A,P,\alpha)}^{(\cup)} : A \rtimes_{\alpha}^a P \rightarrow A \rtimes_{\alpha}^{a,(\cup)} P$ is an isomorphism. In addition, $A \rtimes_{\alpha}^a P$ and $A \rtimes_{\alpha,r}^a P$ can be canonically identified with the ordinary full and reduced crossed product by the group P . The reason is that a group does not have any proper (right) ideals, so that both the families \mathcal{J} and $\mathcal{J}^{(\cup)}$ coincide with the trivial family $\{P, \emptyset\}$ in case P is a group.

As we have already mentioned, our construction of semigroup C^* -algebras extends the one presented by A. Nica in [Ni]. Let us now explain in detail why this is the case:

A. Nica considers positive cones in so-called quasi-lattice ordered groups. If we reformulate A. Nica's conditions in terms of right ideals, then a quasi-lattice ordered group is a pair (G, P) consisting of a (discrete) subsemigroup P of a (discrete) group G such that $P \cap P^{-1} = \{e\}$ where e is the unit element in G , and for every $n \geq 1$ and elements $x_1, \dots, x_n \in G$,

$$(14) \quad P \cap \bigcap_{i=1}^n (x_i \cdot P) \text{ is either empty or of the form } pP \text{ for some } p \in P.$$

Note that for x in G , we set

$$(15) \quad x \cdot P := \{xp : p \in P\} \subseteq G.$$

Comparing this notation with ours from (1), we obtain that for every p, q in P , $q^{-1}pP$ in our notation (1) is the same as $((q^{-1}p) \cdot P) \cap P$ in notation (15). More generally (proceeding inductively on n), we have for all $p_1, \dots, p_n, q_1, \dots, q_n$ in P that $q_1^{-1}p_1 \cdots q_n^{-1}p_n P$ in notation (1) coincides with $P \cap (q_1^{-1}p_1) \cdot P \cap \cdots \cap (q_n^{-1}p_n) \cdot P$ in notation (15). Therefore, for such a semigroup P in a quasi-lattice ordered group (G, P) , the family \mathcal{J} is simply given by

$$(16) \quad \mathcal{J} = \{pP : p \in P\} \cup \{\emptyset\}.$$

In other words, the family \mathcal{J} consists of the empty set and all principal right ideals of P . With this observation, it is now easy to identify A. Nica's construction with ours:

First of all, our definition of the reduced semigroup C*-algebra $C_r^*(P)$ is exactly the same as A. Nica's (see [Ni], § 2.4; A. Nica denotes his reduced semigroup C*-algebra by $\mathcal{W}(G, P)$).

Let us now treat the full versions. A. Nica defines the full semigroup C*-algebra of P (or of the pair (G, P)) as the universal C*-algebra for covariant representations of P by isometries. He denotes this C*-algebra by $C^*(G, P)$. To be more precise, this means that $C^*(G, P)$ is the universal C*-algebra generated by isometries $\{v(p): p \in P\}$ subject to the relations

$$\begin{aligned} \text{I}_{\text{Nica}}. \quad & v(p)v(q) = v(pq) \\ \text{II}_{\text{Nica}}. \quad & v(p)v(p)^*v(q)v(q)^* = \begin{cases} v(r)v(r)^* & \text{if } pP \cap qP = rP \text{ for some } r \in P \\ 0 & \text{if } pP \cap qP = \emptyset \end{cases} \end{aligned}$$

for all p, q in P . Note that by condition (14), there are only these two possibilities $pP \cap qP = rP$ for some $r \in P$ or $pP \cap qP = \emptyset$.

Now we can construct mutually inverse homomorphisms $C^*(P) \rightleftharpoons C^*(G, P)$ as follows: Send $C^*(P) \ni v_p$ to $v(p) \in C^*(G, P)$ and $C^*(P) \ni e_X$ to $0 \in C^*(G, P)$ if $X = \emptyset$ and to $v(p)v(p)^*$ if $X = pP$ (compare (16)). Such a homomorphism $C^*(P) \rightarrow C^*(G, P)$ exists as relation I.(i) is exactly relation I_{Nica} and relation I.(ii) is satisfied as $v_p e_{qP} v_p^* \mapsto v(p)v(q)v(q)^*v(p)^* \stackrel{\text{I}_{\text{Nica}}}{=} v(pq)v(pq)^*$ and $e_{pP} \mapsto v(pq)v(pq)^*$. Moreover, relations II.(i) and II.(ii) are obviously satisfied, and relation II.(iii) corresponds precisely to relation II_{Nica} . For the homomorphism in the reverse direction, set $C^*(P) \ni v_p \leftarrow v(p) \in C^*(G, P)$. Such a homomorphism exists because relation I_{Nica} is relation I.(i), and we have in $C^*(P)$

$$v_p v_p^* v_q v_q^* \stackrel{\text{II.(i)}}{=} v_p e_{pP} v_p^* v_q e_{qP} v_q^* \stackrel{\text{I.(ii)}}{=} e_{pP} e_{qP} = e_{[pP \cap qP]}.$$

If $pP \cap qP$ is of the form rP for some r in P , then $e_{pP \cap qP} = e_{rP} = v_r e_{pP} v_r^* = v_r v_r^*$, and if $pP \cap qP = \emptyset$, then $e_{[pP \cap qP]} = e_\emptyset \stackrel{\text{II.(ii)}}{=} 0$. Therefore, relation II_{Nica} is satisfied. Hence we have seen that $C^*(P)$ and $C^*(G, P)$ are canonically isomorphic. Moreover, we will also see in Corollary 2.29 that if P is the positive cone in a quasi-lattice ordered group, then the canonical homomorphism $\pi^{(\cup)} : C^*(P) \rightarrow C^{*(\cup)}(P)$ is an isomorphism.

So for the special semigroups which A. Nica considers, our constructions indeed coincide with A. Nica's. We refer the reader to [Ni], Sections 1 and 5 for concrete examples already discussed by A. Nica.

Furthermore, let us compare our construction with the one in [C-D-L]. Given a ring of integers R in a number field, the Toeplitz algebra $\mathfrak{T}[R]$ is defined as the universal C*-algebra generated by

$$\begin{aligned} & \text{unitaries } \{u^b: b \in R\}, \\ & \text{isometries } \{s_a: a \in R^\times = R \setminus \{0\}\} \\ & \text{and projections } \{e_I: (0) \neq I \triangleleft R\} \end{aligned}$$

subject to the relations

$$\begin{aligned}
(17) \quad & u^b s_a u^d s_c = u^{b+ad} s_{ac} \\
(18) \quad & e_{I \cap J} = e_I \cdot e_J, \quad e_R = 1 \\
(19) \quad & s_a e_I s_a^* = e_{aI} \\
(20) \quad & u^b e_I u^{-b} = e_I \text{ if } b \in I \text{ and } u^b e_I u^{-b} \perp e_I \text{ if } b \notin I.
\end{aligned}$$

Alternatively, we can consider the $ax + b$ -semigroup over the ring of integers R . It is given by $R \rtimes R^\times = \{(b, a): b \in R, a \in R^\times\}$ where $R^\times = R \setminus \{0\}$, and the binary operation is defined by $(b, a)(d, c) = (b + ad, ac)$. Since R is an integral domain, this semigroup $R \rtimes R^\times$ is left cancellative. So we can apply our construction and consider the semigroup C^* -algebra $C^*(R \rtimes R^\times)$.

Our goal is to show that $C^*(R \rtimes R^\times)$ and $\mathfrak{T}[R]$ are canonically isomorphic. To see this, we first make two observations:

The relations (18) and (20) may be replaced by the stronger relations

$$\begin{aligned}
(21) \quad & e_R = 1 \\
(22) \quad & u^b e_I u^{-b} = e_I \text{ for all } b \in I \\
(23) \quad & u^{b_1} e_{I_1} u^{-b_1} u^{b_2} e_{I_2} u^{-b_2} = \begin{cases} u^d e_{I_1 \cap I_2} u^{-d} & \text{if } (b_1 + I_1) \cap (b_2 + I_2) = d + I_1 \cap I_2 \\ 0 & \text{if } (b_1 + I_1) \cap (b_2 + I_2) = \emptyset. \end{cases}
\end{aligned}$$

First of all, it is easy to see that the two cases which appear in (23) are the only possible cases. To see that the relations (17), (19), (21)–(23) are actually equivalent to the relations (17) – (20), we have to prove that the relations (17) – (20) imply (23). The remaining implications are obvious. Now, if $(b_1 + I_1) \cap (b_2 + I_2) = \emptyset$, then $-b_1 + b_2$ does not lie in $I_1 + I_2$. Hence

$$u^{b_1} e_{I_1} u^{-b_1} u^{b_2} e_{I_2} u^{-b_2} \stackrel{(18)}{=} u^{b_1} e_{I_1} \underbrace{e_{I_1 + I_2} u^{-b_1 + b_2} e_{I_1 + I_2}}_{= 0 \text{ by (20)}} e_{I_2} u^{-b_2} = 0.$$

If $(b_1 + I_1) \cap (b_2 + I_2) = d + I_1 \cap I_2$, then we can find elements $r_1, r_2 \in R$ so that $d = b_1 + r_1 = b_2 + r_2 \Rightarrow -b_1 + b_2 = r_1 - r_2$. We conclude that

$$\begin{aligned}
& u^{b_1} e_{I_1} u^{-b_1} u^{b_2} e_{I_2} u^{-b_2} = u^{b_1} e_{I_1} u^{r_1} u^{-r_2} e_{I_2} u^{-b_2} \\
& \stackrel{(20)}{=} u^{b_1} u^{r_1} e_{I_1} e_{I_2} u^{-r_2} u^{b_2} \stackrel{(17), (18)}{=} u^d e_{[I_1 \cap I_2]} u^{-d}.
\end{aligned}$$

Moreover, using the fact that R is a Dedekind domain (the definition of a Dedekind domain is for instance given in [Neu], Chapter I, Definition (3.2)), we can deduce that every ideal $(0) \neq I \triangleleft R$ is of the form $I = ((c^{-1}a) \cdot R) \cap R$ for some $a, c \in R^\times$. (Here $(\cdot)^{-1}$ stands for the inverse in the multiplicative group of the quotient field of R .) A proof of this observation is given in [C-D-L], Lemma 4.15. Here is an alternative proof: Since R is a Dedekind domain, we can find non-zero prime ideals P_1, \dots, P_n so that $I = P_1^{\nu_1} \dots P_n^{\nu_n}$. By strong approximation (see [Bour2], Chapitre VII, § 2.4, Proposition 2), there are $a, c \in R^\times$ such that

$$aR = P_1^{\nu_1} \dots P_n^{\nu_n} I_a \text{ for some ideal } I_a \text{ which is coprime to } P_1, \dots, P_n$$

and

$$cR = I_a I_c \text{ for some ideal } I_c \text{ which is coprime to } I_a \text{ and } P_1, \dots, P_n.$$

We then have

$$(c^{-1}a) \cdot R = P_1^{\nu_1} \cdots P_n^{\nu_n} (I_c)^{-1}$$

so that

$$((c^{-1}a) \cdot R) \cap R = P_1^{\nu_1} \cdots P_n^{\nu_n} = I.$$

This proof shows that in an arbitrary Dedekind domain R , every ideal $(0) \neq I \triangleleft R$ is of the form $I = ((c^{-1}a) \cdot R) \cap R$. As $((c^{-1}a) \cdot R) \cap R = c^{-1}(aR)$ where on the right hand side, c^{-1} stands for pre-image (under left multiplication with c), it follows that for the semigroup $R \rtimes R^\times$, the family \mathcal{J} is given by

$$\mathcal{J} = \{(b+I) \times I^\times : b \in R, (0) \neq I \triangleleft R\} \cup \{\emptyset\},$$

where $I^\times = I \cap R^\times = I \setminus \{0\}$. Again, this not only holds for rings of integers, but for arbitrary Dedekind domains.

We can now construct mutually inverse homomorphisms $C^*(R \rtimes R^\times) \rightleftharpoons \mathfrak{T}[R]$ by setting

$$v_{(b,a)} \mapsto u^b s_a, e_{(b+I) \times I^\times} \mapsto u^b e_I u^{-b}, e_\emptyset \mapsto 0$$

and

$$v_{(b,1)} \leftarrow u^b, v_{(0,a)} \leftarrow s_a, e_{I \times I^\times} \leftarrow e_I.$$

To see that these homomorphisms really exist, we have to compare the relations from Definition 2.2 defining $C^*(R \rtimes R^\times)$ with the relations (17), (19) and (21)–(23). It is easy to see that

- relation I.(i) corresponds to relation (17),
- relation I.(ii) for $p = (0, a) \in R \rtimes R^\times$ corresponds to relation (19),
- relation II.(i) is relation (21),
- relation I.(ii) for $p = (b, 1) \in R \rtimes R^\times$ is relation (22)
- and relation II.(iii), together with relation II.(ii), is relation (23).

This proves that $C^*(R \rtimes R^\times)$ and $\mathfrak{T}[R]$ are canonically isomorphic.

2.5. Functoriality. At this point, we would like to address the question of functoriality: Given a homomorphism $\varphi : P \rightarrow Q$ between left cancellative semigroups, does φ induce a homomorphism of the semigroup C*-algebras by the formula $v_p \mapsto v_{\varphi(p)}$?

It is not clear what the answer to this question in general is because the assignment $v_p \mapsto v_{\varphi(p)}$ has to be compatible with the extra relations we have built into our constructions. One thing that is clear is that a homomorphism $C^*(P) \rightarrow C^*(Q)$ is uniquely determined by the requirement that v_p is sent to $v_{\varphi(p)}$ for all p in P . The reason is that $C^*(P)$ is generated as a C*-algebra by the isometries v_p (see Corollary 2.10). However, for special semigroups, namely $ax + b$ -semigroups over integral domains, we can say more about functoriality.

We consider the following setting: Let R be an integral domain, i.e. a commutative ring with unit but without zero-divisors. As we did before in the case of rings of integers, we can form the $ax + b$ -semigroup P_R over R . To be more precise, P_R is

the semidirect product $R \rtimes R^\times$, where $R^\times = R \setminus \{0\}$ acts multiplicatively on R . This means that $P_R = \{(b, a) : b \in R, a \in R^\times\}$ and the binary operation is given by $(b, a)(d, c) = (b + ad, ac)$. P_R is left cancellative because R has no zero-divisors. Thus we can form the semigroup C^* -algebra $C^*(P_R)$. Let us describe the family \mathcal{J}_{P_R} given by (5) for this semigroup P_R . Given an ideal I of R , we denote its image under left multiplication by $a \in R^\times$ by aI and its pre-image under left multiplication with $a \in R^\times$ by $a^{-1}I$, i.e. $aI = \{ar : r \in I\}$ and $a^{-1}I = \{r \in R : ar \in I\}$. Let $\mathcal{I}(R)$ be the smallest family of ideals of R which contains R , which is closed under left multiplications as well as pre-images under left multiplications, i.e. $a \in R^\times, I \in \mathcal{I}(R) \Rightarrow aI, a^{-1}I \in \mathcal{I}(R)$, and finite intersections, i.e. $I, J \in \mathcal{I}(R) \Rightarrow I \cap J \in \mathcal{I}(R)$. By definition, we have

$$\mathcal{I}(R) = \left\{ \bigcap_{j=1}^N (c_{j,1})^{-1} a_{j,1} \cdots (c_{j,n_j})^{-1} a_{j,n_j} R : N, n_j \in \mathbb{Z}_{>0}; a_{j,k}, c_{j,k} \in R^\times \right\}.$$

We then have

$$\mathcal{J}_{P_R} = \{(b + I) \times I^\times : b \in R, I \in \mathcal{I}(R)\} \cup \{\emptyset\},$$

where $I^\times = I \cap R^\times = I \setminus \{0\}$.

Now assume that S is another integral domain, and let P_S be the $ax + b$ -semigroup over S . Moreover, let ϕ be a ring homomorphism $R \rightarrow S$. If ϕ is injective, it induces a semigroup homomorphism $\varphi : P_R \rightarrow P_S$ which sends $P_R \ni (b, a)$ to $(\phi(b), \phi(a)) \in P_S$. Extending the functorial results on Toeplitz algebras associated with rings of integers in number fields from [C-D-L], Proposition 3.2, we show that there exists a homomorphism $C^*(P_R) \rightarrow C^*(P_S)$ sending v_p to $v_{\varphi(p)}$ for every $p \in P$ if φ comes from a ring monomorphism ϕ such that the quotient $S/\phi(R)$ (in the category of $\phi(R)$ -modules) is a flat $\phi(R)$ -module.

Lemma 2.18. *Assume that for all ideals I and J of R which lie in $\mathcal{I}(R)$, we have*

- (a) $(\phi(I)S) \cap \phi(R) = \phi(I)$
- (b) $\phi(I)S \cap \phi(J)S = \phi(I \cap J)S$.

Then there exists a homomorphism $C^(P_R) \rightarrow C^*(P_S)$ sending v_p to $v_{\varphi(p)}$ for every $p \in P_R$.*

By $\phi(I)S$, we mean the ideal of S generated by $\phi(I)$.

Proof. By universal property of $C^*(P_R)$, there exists a homomorphism $C^*(P_R) \rightarrow C^*(P_S)$ sending $C^*(P_R) \ni v_p$ to $v_{\varphi(p)} \in C^*(P_S)$ for every $p \in P_R$ and $C^*(P_R) \ni e_{[(b+I) \times I^\times]}$ to $e_{[(\phi(b)+\phi(I)S) \times (\phi(I)S)^\times]} \in C^*(P_S)$ for every $b \in R, I \in \mathcal{I}(R)$. To see this, we first of all have to prove that for every $(b + I) \times I^\times \in \mathcal{J}_{P_R}$, the right ideal $(\phi(b) + \phi(I)S) \times (\phi(I)S)^\times$ lies in \mathcal{J}_{P_S} . It suffices to show that for every $I \in \mathcal{I}(R)$, the ideal $\phi(I)S$ lies in $\mathcal{I}(S)$, where

$$\mathcal{I}(S) = \left\{ \bigcap_{j=1}^N (c_{j,1})^{-1} a_{j,1} \cdots (c_{j,n_j})^{-1} a_{j,n_j} S : N, n_j \in \mathbb{Z}_{>0}; a_{j,k}, c_{j,k} \in S^\times \right\}.$$

All we have to prove is that for all $a, c \in R^\times$ and every $I \in \mathcal{I}(R)$, we have

$$(24) \quad \phi(aI)S = \phi(a)(\phi(I)S),$$

$$(25) \quad \phi(c^{-1}I)S = \phi(c)^{-1}(\phi(I)S).$$

(24) is obviously true. For (25), we observe that

$$\begin{aligned} \phi(c)(\phi(c^{-1}I)S) &= \phi(c(c^{-1}I))S = \phi(I \cap cR)S \\ &\stackrel{(b)}{=} \phi(I)S \cap \phi(cR)S = \phi(I)S \cap \phi(c)S = \phi(c)(\phi(c)^{-1}(\phi(I)S)). \end{aligned}$$

Applying $\phi(c)^{-1}$ to both sides of this equation yields $\phi(c^{-1}I)S = \phi(c)^{-1}(\phi(I)S)$, as desired.

Moreover, we have to check that the map

$$\mathcal{J}_{P_R} \ni (b + I) \times I^\times \mapsto (\phi(b) + \phi(I)S) \times (\phi(I)S)^\times \in \mathcal{J}_{P_S}$$

is compatible with left multiplications, taking pre-images under left multiplications and finite intersections. (24) and (25) imply compatibility with left multiplications and taking pre-images under left multiplications. It remains to prove compatibility with finite intersections. More precisely, we have to show that if

$$(26) \quad ((b + I) \times I^\times) \cap ((d + J) \times J^\times) = \emptyset,$$

then

$$(27) \quad ((\phi(b) + \phi(I)S) \times (\phi(I)S)^\times) \cap ((\phi(d) + \phi(J)S) \times (\phi(J)S)^\times) = \emptyset,$$

and if

$$(28) \quad ((b + I) \times I^\times) \cap ((d + J) \times J^\times) = (r + I \cap J) \times (I \cap J)^\times \text{ for some } r \in R,$$

then

$$\begin{aligned} (29) \quad & ((\phi(b) + \phi(I)S) \times (\phi(I)S)^\times) \cap ((\phi(d) + \phi(J)S) \times (\phi(J)S)^\times) \\ &= (\phi(r) + \phi(I \cap J)S) \times (\phi(I \cap J)S)^\times. \end{aligned}$$

Now (26) holds if and only if $(b + I) \cap (d + J) = \emptyset \Leftrightarrow b - d \notin I + J$. If the difference $b - d$ does not lie in $I + J$, then $\phi(b) - \phi(d)$ does not lie in

$$\phi(I + J) \stackrel{(a)}{=} \phi(I + J)S \cap \phi(R) = (\phi(I)S + \phi(J)S) \cap \phi(R).$$

Hence $\phi(b) - \phi(d)$ does not lie in $\phi(I)S + \phi(J)S$. This implies $(\phi(b) + \phi(I)S) \cap (\phi(d) + \phi(J)S) = \emptyset$, and (27) follows. Moreover, (28) holds if and only if $(b + I) \cap (d + J) = r + I \cap J \Leftrightarrow r \in (b + I) \cap (d + J)$ for some $r \in R$. If r lies in $b + I$, then $\phi(r)$ lies in $\phi(b) + \phi(I)S$. Similarly, $\phi(r)$ lies in $\phi(d) + \phi(J)S$ if r lies in $d + J$. Thus if (28) holds, then $\phi(r)$ lies in $(\phi(b) + \phi(I)S) \cap (\phi(d) + \phi(J)S)$. This implies

$$(\phi(b) + \phi(I)S) \cap (\phi(d) + \phi(J)S) = \phi(r) + \phi(I)S \cap \phi(J)S \stackrel{(b)}{=} \phi(r) + \phi(I \cap J)S.$$

This implies (29). \square

Corollary 2.19. *Assume that $\phi : R \rightarrow S$ is an inclusion of integral domains such that the quotient $S/\phi(R)$ of the $\phi(R)$ -module S by the $\phi(R)$ -module $\phi(R)$ (in the category of $\phi(R)$ -modules) is a flat $\phi(R)$ -module. Let P_R and P_S be the $ax + b$ -semigroups over R and S , respectively, and let $\varphi : P_R \rightarrow P_S$ be the semigroup homomorphism induced by ϕ . Then there exists a homomorphism $\Phi : C^*(P_R) \rightarrow C^*(P_S)$ sending $C^*(P_R) \ni v_p$ to $v_{\varphi(p)} \in C^*(P_S)$.*

We remark that the condition of flatness already appears in [C-D-L], Lemma 3.1.

Proof. If $S/\phi(R)$, the quotient in the category of $\phi(R)$ -modules of S by $\phi(R)$, is a flat $\phi(R)$ -module, then S itself is a flat $\phi(R)$ -module by [Bour1], Chapitre I, § 2.5 Proposition 5 using that $\phi(R)$ is flat as a module over itself. Therefore, conditions (a) and (b) from the previous lemma are satisfied, see for instance [Bour1], Chapitre I, § 2.6 Proposition 6 and Corollaire (to Proposition 7). \square

2.6. Comparison of universal C*-algebras. In the last part of this section, let us compare the universal C*-algebras $C^*(P)$ and $C^{*(\cup)}(P)$. Our goal is to find out under which conditions the canonical homomorphism $\pi^{(\cup)} : C^*(P) \rightarrow C^{*(\cup)}(P)$ is an isomorphism. It will be possible to give a criterion in terms of the constructible right ideals of P . As a first step, we take a look at the commutative sub-C*-algebras $D(P)$ and $D^{(\cup)}(P)$ of $C^*(P)$ and $C^{*(\cup)}(P)$. Our investigations will also involve the commutative sub-C*-algebra $D_r(P)$ of the reduced semigroup C*-algebra.

Lemma 2.20. *Let D be a unital C*-algebra generated by commuting projections $\{f_i\}_{i \in I}$. For a non-empty finite set $F \subseteq I$ and a non-empty subset $F' \subseteq F$, define the projection $e(F', F)$ as*

$$e(F', F) := \left(\prod_{i \in F'} f_i \right) \cdot \left(\prod_{i \in F \setminus F'} (1 - f_i) \right).$$

Then, given a C-algebra C , a homomorphism $\varphi : D \rightarrow C$ is injective if and only if for every non-empty finite subset $F \subseteq I$ and $\emptyset \neq F' \subseteq F$ as above,*

$$(30) \quad \varphi(e(F', F)) = 0 \text{ in } C \text{ implies } e(F', F) = 0 \text{ in } D.$$

Proof. If φ is injective, then certainly $\varphi(e(F', F)) = 0$ must imply $e(F', F) = 0$. To prove the reverse implication, we set $D_F := C^*(\{f_i : i \in F\}) \subseteq D$ for every non-empty finite subset $F \subseteq I$. The non-empty finite subsets of I are ordered by inclusion, and we obviously have

$$D = \overline{\bigcup_{\emptyset \neq F \subseteq I \text{ finite}} D_F}.$$

So it remains to prove that if condition (30) holds for a non-empty finite subset $F \subseteq I$, then $\varphi|_{D_F}$ is injective.

But since the projections $\{f_i : i \in F\}$ commute, it is clear that the projections $e(F', F)$, $\emptyset \neq F' \subseteq F$ are pairwise orthogonal. This implies that

$$D_F = \bigoplus_{\emptyset \neq F' \subseteq F} \mathbb{C} \cdot e(F', F).$$

Hence it follows that $\varphi|_{D_F}$ is injective if and only if (30) holds for every non-empty subset F' of F . \square

As a next step, we work out how the projections $e(F', F)$ look like in the following situation: Let $D = D^{(\cup)}(P)$, $I = \mathcal{J}^{(\cup)}$ and for every $X \in \mathcal{J}^{(\cup)}$, set $f_X := e_X \in C^{*(\cup)}(P)$ (see Definition 2.4).

Lemma 2.21. *For every non-empty finite subset $F \subseteq \mathcal{J}^{(\cup)}$ and every $\emptyset \neq F' \subseteq F$, there exist $X, Y \in \mathcal{J}^{(\cup)}$ with $Y \subseteq X$ such that $e(F', F) = e_X - e_Y$.*

Proof. Let us proceed inductively on $|F|$. The starting point $|F| = 1$ is trivial. We assume that the claim is proven whenever $|F| = n$. Let F be a finite subset of $\mathcal{J}^{(\cup)}$ with $|F| = n + 1$. If $F' = F$ then our assertion obviously follows from relation $\Pi^{(\cup)}.(\text{iii})$. If $\emptyset \neq F' \subsetneq F$, then we can find a subset F_n of $\mathcal{J}^{(\cup)}$ with $|F_n| = n$ and $F' \subseteq F_n \subseteq F$. Let $F = F_n \cup \{X_{n+1}\}$. We know by induction hypothesis that there exist $X_n, Y_n \in \mathcal{J}^{(\cup)}$ with $Y_n \subseteq X_n$ such that $e(F', F_n) = e_{X_n} - e_{Y_n}$. Therefore,

$$\begin{aligned} e(F', F) &= e(F', F_n)(1 - e_{X_{n+1}}) = (e_{X_n} - e_{Y_n})(1 - e_{X_{n+1}}) \\ &\stackrel{\Pi^{(\cup)}.(\text{iii})}{=} e_{X_n} - e_{Y_n} - e_{[X_n \cap X_{n+1}]} + e_{[Y_n \cap X_{n+1}]} \stackrel{\Pi^{(\cup)}.(\text{iv})}{=} e_{X_n} - e_{[Y_n \cup (X_n \cap X_{n+1})]}. \end{aligned}$$

Set $X = X_n$, $Y = Y_n \cup (X_n \cap X_{n+1})$ and we are done. \square

Corollary 2.22. $\lambda^{(\cup)}|_{D^{(\cup)}(P)} : D^{(\cup)}(P) \rightarrow D_r(P)$ is an isomorphism.

Proof. It is clear that $\lambda^{(\cup)}|_{D^{(\cup)}(P)}$ is surjective, thus it remains to prove injectivity. We want to apply Lemma 2.20 to $D = D^{(\cup)}(P) = C^*(\{e_X : X \in \mathcal{J}^{(\cup)}\})$, $C = D_r(P)$ and $\varphi = \lambda^{(\cup)}|_{D^{(\cup)}(P)}$. For a non-empty finite subset $F \subseteq \mathcal{J}^{(\cup)}$ and $\emptyset \neq F' \subseteq F$, Lemma 2.21 tells us that there are $X, Y \in \mathcal{J}^{(\cup)}$ with $Y \subseteq X$ such that $e(F', F) = e_X - e_Y$. Now $\lambda^{(\cup)}(e_X - e_Y) = E_X - E_Y$, and $E_X - E_Y$ vanishes as an operator on $\ell^2(P)$ if and only if $X = Y$. But $X = Y$ obviously implies $e(F', F) = e_X - e_Y = 0$ in $D^{(\cup)}(P)$. Therefore, Lemma 2.20 implies that $\lambda^{(\cup)}|_{D^{(\cup)}(P)}$ must be injective. \square

Corollary 2.23. *Given two left cancellative semigroups P and Q , we can identify $C^{*(\cup)}(P \times Q)$ with $C^{*(\cup)}(P) \otimes_{\max} C^{*(\cup)}(Q)$ via a homomorphism sending $v_{(p,q)}$ to $v_p \otimes v_q$ for every $p \in P$ and $q \in Q$.*

Proof. As explained in Remark 2.17, all we have to do is to construct a homomorphism $D^{(\cup)}(P \times Q) \rightarrow C^{*(\cup)}(P) \otimes_{\max} C^{*(\cup)}(Q)$ which sends for all $X \in \mathcal{J}_P$ and $Y \in \mathcal{J}_Q$ the projection $e_{X \times Y}$ to $e_X \otimes e_Y$. But we know by the previous lemma that $D^{(\cup)}(P \times Q) \cong D_r(P \times Q)$, $D^{(\cup)}(P) \cong D_r(P)$ and $D^{(\cup)}(Q) \cong D_r(Q)$. Moreover, the isomorphism $C_r^*(P \times Q) \cong C_r^*(P) \otimes_{\min} C_r^*(Q)$ from Lemma 2.16 obviously identifies $D_r(P \times Q)$ with $D_r(P) \otimes_{\min} D_r(Q)$. Thus the desired homomorphism is given by

$$\begin{aligned} D^{(\cup)}(P \times Q) &\cong D_r(P \times Q) \cong D_r(P) \otimes_{\min} D_r(Q) \cong D_r(P) \otimes_{\max} D_r(Q) \\ &\cong D^{(\cup)}(P) \otimes_{\max} D^{(\cup)}(Q) \rightarrow C^{*(\cup)}(P) \otimes_{\max} C^{*(\cup)}(Q). \end{aligned}$$

\square

Now we come to the main result concluding this circle of ideas.

Proposition 2.24. *The following statements are equivalent:*

- (i) If $X = \bigcup_{j=1}^n X_j$ for $X, X_1, \dots, X_n \in \mathcal{J}$, then $X = X_j$ for some $1 \leq j \leq n$.
- (ii) $\pi^{(\cup)}|_{D(P)} : D(P) \rightarrow D^{(\cup)}(P)$ is an isomorphism.
- (iii) $\pi^{(\cup)} : C^*(P) \rightarrow C^{*(\cup)}(P)$ is an isomorphism.
- (iv) There exists a homomorphism $\Delta^{(\cup)} : C^{*(\cup)}(P) \rightarrow C^{*(\cup)}(P) \otimes_{\max} C^{*(\cup)}(P)$ which sends (for all $p \in P$) v_p to $v_p \otimes v_p$.
- (v) There exists a homomorphism $\Delta_D^{(\cup)} : D^{(\cup)}(P) \rightarrow D^{(\cup)}(P) \otimes_{\max} D^{(\cup)}(P)$ which sends (for all $X \in \mathcal{J}$) e_X to $e_X \otimes e_X$.

Proof. “(i) \Rightarrow (ii)”: Since by Corollary 2.22, $\lambda^{(\cup)}|_{D^{(\cup)}(P)}$ is an isomorphism and because we always have $\lambda = \lambda^{(\cup)} \circ \pi^{(\cup)}$, statement (ii) is equivalent to “ $\lambda|_{D(P)}$ is an isomorphism”. $\lambda|_{D(P)}$ is obviously surjective, so it remains to prove injectivity. We want to apply Lemma 2.20 to $D = D(P)$, $I = \mathcal{J}$, $f_X := e_X \in D(P)$ for $X \in \mathcal{J}$, $C = D_r(P)$ and $\varphi = \lambda|_{D(P)}$. Given a non-empty finite subset $F \subseteq \mathcal{J}$ and $\emptyset \neq F' \subseteq F$, it is immediate that $\lambda(e(F', F)) = E_{[(\bigcap_{X' \in F'} X') \setminus (\bigcup_{Y \in F \setminus F'} Y)]}$ where $E_{[(\bigcap_{X' \in F'} X') \setminus (\bigcup_{Y \in F \setminus F'} Y)]}$ is the orthogonal projection onto the subspace

$$\ell^2 \left(\left(\bigcap_{X' \in F'} X' \right) \setminus \left(\bigcup_{Y \in F \setminus F'} Y \right) \right) \subseteq \ell^2(P).$$

Assume that $\lambda(e(F', F))$ vanishes. Then $X := \bigcap_{X' \in F'} X'$ must be a subset of $\bigcup_{Y \in F \setminus F'} Y$. Now X lies in \mathcal{J} , and $X \subseteq \bigcup_{Y \in F \setminus F'} Y$ implies $X = \bigcup_{Y \in F \setminus F'} (Y \cap X)$. But statement (i) tells us that this can only happen if there exists $Y \in F \setminus F'$ with $Y \cap X = X$, or equivalently, $X \subseteq Y$. Thus $e_X = e_{X \cap Y} \stackrel{II.(iii)}{=} e_X \cdot e_Y$, and we conclude that $e_X(1 - e_Y) = 0$. Hence it follows that

$$e(F', F) = e_X(1 - e_Y) \cdot \prod_{Y \neq Z \in F \setminus F'} (1 - e_Z) = 0.$$

So we have seen that condition (30) holds. Therefore $\lambda|_{D(P)}$ is injective.

“(ii) \Rightarrow (iii)”: This follows from the crossed product descriptions of $C^*(P)$ and $C^{*(\cup)}(P)$ from Lemma 2.14 and the fact that $\pi^{(\cup)}|_{D(P)}$ is P -equivariant with respect to the actions τ and $\tau^{(\cup)}$.

“(iii) \Rightarrow (iv)”: It follows from universal property of $C^*(P)$ that there exists a homomorphism $\Delta : C^*(P) \rightarrow C^*(P) \otimes_{\max} C^*(P)$ which sends v_p to $v_p \otimes v_p \in C^*(P) \odot C^*(P) \subseteq C^*(P) \otimes_{\max} C^*(P)$ and e_X to $e_X \otimes e_X \in C^*(P) \odot C^*(P) \subseteq C^*(P) \otimes_{\max} C^*(P)$ for every $p \in P$ and $X \in \mathcal{J}$. The reason is that relations I and II are obviously valid with $v_p \otimes v_p$ in place of v_p and $e_X \otimes e_X$ in place of e_X . Now set $\Delta^{(\cup)} := ((\pi^{(\cup)})^{-1} \otimes_{\max} (\pi^{(\cup)})^{-1}) \circ \Delta \circ \pi^{(\cup)}$.

“(iv) \Rightarrow (v)”: Just restrict $\Delta^{(\cup)}$ to $D^{(\cup)}(P)$, i.e. set $\Delta_D^{(\cup)} := \Delta^{(\cup)}|_{D^{(\cup)}(P)}$.

“(v) \Rightarrow (i)”: Let D be the sub- $*$ -algebra of $D^{(\cup)}(P)$ generated by the projections $\{e_X : X \in \mathcal{J}^{(\cup)}\}$. By relation II^(u).(iii), the set $\{e_X : X \in \mathcal{J}\}$ is multiplicatively

closed, and by relation $\text{II}^{(\cup)}.\text{(iv)}$, $D = \text{span}(\{e_X: \emptyset \neq X \in \mathcal{J}\})$. Restricting $\Delta_D^{(\cup)}$ to D , we obtain a homomorphism $D \rightarrow D \odot D$ which is determined by $e_X \mapsto e_X \otimes e_X$ for every $X \in \mathcal{J}$. Let us denote this restriction by Δ_D .

We can now deduce from the existence of such a homomorphism Δ_D that the set $\{e_X: \emptyset \neq X \in \mathcal{J}\}$ is a \mathbb{C} -basis of D . As $\{e_X: \emptyset \neq X \in \mathcal{J}\}$ generates D as a \mathbb{C} -vector space, we can always find a subset $\tilde{\mathcal{J}}$ of $\mathcal{J} \setminus \{\emptyset\}$ such that $\{e_X: X \in \tilde{\mathcal{J}}\}$ is a \mathbb{C} -basis for D . It then follows that $\{e_{\tilde{X}} \otimes e_{\tilde{Y}}: \tilde{X}, \tilde{Y} \in \tilde{\mathcal{J}}\}$ is a \mathbb{C} -basis of $D \odot D$.

Now take $\emptyset \neq X \in \mathcal{J}$. We can find finite subsets $\{X_i\} \subseteq \tilde{\mathcal{J}}$ and $\{\alpha_i\} \subseteq \mathbb{C}$ with $e_X = \sum_i \alpha_i e_{X_i}$. Applying Δ_D yields

$$\sum_{i,j} \alpha_i \alpha_j e_{X_i} \otimes e_{X_j} = e_X \otimes e_X = \Delta_D(e_X) = \sum_i \alpha_i \Delta_D(e_{X_i}) = \sum_i \alpha_i e_{X_i} \otimes e_{X_i}.$$

Hence it follows that among the α_i s, there can only be one non-zero coefficient which must be 1. The corresponding vector e_{X_i} must then coincide with e_X . This implies $e_X \in \{e_{\tilde{X}}: \tilde{X} \in \tilde{\mathcal{J}}\}$, i.e. $\{e_X: \emptyset \neq X \in \mathcal{J}\} = \{e_{\tilde{X}}: \tilde{X} \in \tilde{\mathcal{J}}\}$ is a \mathbb{C} -basis of D .

Now assume that there are $X, X_1, \dots, X_n \in \mathcal{J}$ with $X = \bigcup_{j=1}^n X_j$. We necessarily have $X_j \subseteq X$ for all $1 \leq j \leq n$. Moreover, $X_j \subsetneq X$ implies $e_{X_j} \leq e_X$ because $\lambda(e_{X_j}) = E_{X_j} \leq E_X = \lambda(e_X)$ as concrete operators on $\ell^2(P)$. Using relation $\text{II}^{(\cup)}.\text{(iv)}$, we obtain from $X = \bigcup_{j=1}^n X_j$ that

$$(31) \quad e_X = \sum_{\emptyset \neq F \subseteq \{1, \dots, n\}} (-1)^{|F|+1} e_{[\bigcap_{j \in F} X_j]}$$

holds in D . But if all the X_j s ($1 \leq j \leq n$) are strictly contained in X , then (31) would give a non-trivial relation among e_X and those projections $e_{[\bigcap_{j \in F} X_j]}$, $\emptyset \neq F \subseteq \{1, \dots, n\}$ which are non-zero. But this contradicts our observation that $\{e_X: \emptyset \neq X \in \mathcal{J}\}$ is a \mathbb{C} -basis of D . Hence we conclude that one of the X_j s must be equal to X . This proves (i). \square

Remark 2.25. This proposition does not really have much to do with semigroups. It actually is a statement about families of subsets of a fixed set and a projection-valued spectral measure defined on this family.

Definition 2.26. We call \mathcal{J} independent (or we also say that the constructible right ideals of P are independent) if the right ideals in \mathcal{J} satisfy (i) from Proposition 2.24.

Note that statement (i) is equivalent to the following one: For all X, X_1, \dots, X_n in \mathcal{J} such that X_1, \dots, X_n are proper subsets of X ($X_i \subsetneq X$ for all $1 \leq i \leq n$), then $\bigcup_{i=1}^n X_i$ must be a proper subset of X ($\bigcup_{i=1}^n X_i \subsetneq X$).

Corollary 2.27. The constructible right ideals of P are independent if and only if the restriction of the left regular representation to the commutative sub-C*-algebra $D(P)$ of the full semigroup C*-algebra $C^*(P)$ is an isomorphism.

Proof. This follows immediately from the equivalence of (i) and (ii) in Proposition 2.24 and from Corollary 2.22. \square

An immediate question that comes to mind after Proposition 2.24 is which semigroups have independent constructible right ideals. The general answer is not known to the author. But we can discuss two particular cases:

Lemma 2.28. *The constructible right ideals of the positive cone in a quasi-lattice ordered group are independent.*

Proof. This follows immediately from the observation that for a semigroup P which is the positive cone in a quasi-lattice ordered group, the family \mathcal{J} consists of the empty set and all principal right ideals of P , see (16). \square

As an immediate consequence of this lemma and Proposition 2.24, we obtain

Corollary 2.29. *If P is the positive cone in a quasi-lattice ordered group, then the canonical homomorphism $\pi^{(\cup)} : C^*(P) \rightarrow C^{*(\cup)}(P)$ is an isomorphism.*

Another class of semigroups with independent constructible right ideals is given as follows:

Lemma 2.30. *Let R be a Dedekind domain. Then the constructible right ideals of the $ax + b$ -semigroup P_R over R are independent.*

Proof. Recall that we have shown above when we identified Toeplitz algebras of rings of integers with full semigroup C^* -algebras of the corresponding $ax + b$ -semigroups that

$$\mathcal{J}_{P_R} = \{(b + I) \times I^\times : b \in R, (0) \neq I \triangleleft R\} \cup \{\emptyset\}.$$

Assume that we have

$$(b + I) \times I^\times = \bigcup_{j=1}^n (b_j + I_j) \times I_j^\times$$

with $(b_j + I_j) \times I_j^\times \subsetneq (b + I) \times I^\times$ for all $1 \leq j \leq n$. Then it follows that $I = \bigcup_{j=1}^n I_j$ with $I_j \subsetneq I$ for all $1 \leq j \leq n$.

Because R is a Dedekind domain, we can find non-zero prime ideals P_1, \dots, P_N of R so that

$$I = P_1^{\nu_1} \cdots P_M^{\nu_M} \text{ for some } M \leq N \text{ and } \nu_1, \dots, \nu_M > 0$$

and

$$I_j = P_1^{\nu_{1,j}} \cdots P_M^{\nu_{M,j}} \cdots P_N^{\nu_{N,j}} \text{ for some } \nu_{i,j} \geq 0 \text{ with } \nu_{i,j} \geq \nu_i \text{ for all } 1 \leq i \leq M.$$

By strong approximation (see [Bour2], Chapitre VII, § 2.4, Proposition 2), there exists $x \in R$ with the properties

$$(*) \quad x \in P_i^{\nu_i} \setminus P_i^{\nu_i+1} \text{ for all } 1 \leq i \leq M$$

(**) $x \notin P_i$ for all $M < i \leq N$.

(*) implies that x lies in I . But x does not lie in I_j for any $1 \leq j \leq n$: If $I_j \subseteq P_i$ for some $M < i \leq N$, then (**) implies that $x \notin I_j \subseteq P_i$. If I_j is coprime to P_i for all $M < i \leq N$ (i.e. $\nu_{i,j} = 0$ for all $M < i \leq N$), then $I_j \subsetneq I$ implies $\nu_{i,j} > \nu_i$ for some $1 \leq i \leq M$. So (*) implies that $x \notin I_j \subseteq P_i^{\nu_{i,j}} \subseteq P_i^{\nu_i+1}$. But this implies that $I \subsetneq \bigcup_{j=1}^n I_j$ which contradicts our assumption. \square

In particular, the constructible right ideals of the $ax + b$ -semigroup P_R over the ring of integers R in a number field are independent. So by Corollary 2.27, the left regular representation restricted to the commutative sub-C*-algebra $D(P_R)$ is an isomorphism. This explains Corollary 4.16 in [C-D-L] ($\mathfrak{T}[R]$ in [C-D-L] is canonically isomorphic to $C^*(P_R)$ as explained above, and \mathfrak{T} in [C-D-L] is $C_r^*(P_R)$).

Remark 2.31. In the proof of Lemma 2.30, we have just shown that whenever given non-zero ideals I, I_1, \dots, I_n of a Dedekind domain R such that I_1, \dots, I_n are proper subsets of I , then $\bigcup_{i=1}^n I_i$ is a proper subset of I . This means that already the non-zero ideals of a Dedekind domain are independent.

3. A VARIANT OF OUR CONSTRUCTION FOR SUBSEMIGROUPS OF GROUPS

Given a subsemigroup of a group, let us now modify our construction of full semigroup C*-algebras. We impose extra relations besides the ones from Definition 2.2. These relations are motivated by the following

Lemma 3.1. *Let P be a subsemigroup of a group G . Given $p_1, q_1, \dots, p_m, q_m$ in P with $p_1^{-1}q_1 \cdots p_m^{-1}q_m = e$ in G , then $V_{p_1}^* V_{q_1} \cdots V_{p_m}^* V_{q_m} = E_{[q_m^{-1}p_m \cdots q_1^{-1}p_1 P]}$ in $C_r^*(P)$.*

Proof. For $x \in P$, we have $E_{[q_m^{-1}p_m \cdots q_1^{-1}p_1 P]} \varepsilon_x = \varepsilon_x$ if $x \in p_m^{-1}p_m \cdots q_1^{-1}p_1 P$ and $E_{[q_m^{-1}p_m \cdots q_1^{-1}p_1 P]} \varepsilon_x = 0$ if $x \notin q_m^{-1}p_m \cdots q_1^{-1}p_1 P$. A direct computation yields that $(V_{p_1}^* V_{q_1} \cdots V_{p_m}^* V_{q_m})(\varepsilon_x) \neq 0$ if and only if x lies in $q_m^{-1}p_m \cdots q_1^{-1}p_1 P$, and in this case, we have $(V_{p_1}^* V_{q_1} \cdots V_{p_m}^* V_{q_m})(\varepsilon_x) = \varepsilon_{p_1^{-1}q_1 \cdots p_m^{-1}q_m x} = \varepsilon_x$. \square

Definition 3.2. *Let P be a subsemigroup of a group G . We let $C_s^*(P)$ be the universal C*-algebra generated by isometries $\{v_p: p \in P\}$ and projections $\{e_X: X \in \mathcal{J}\}$ satisfying the following relations:*

- I. $v_{pq} = v_p v_q$,
- II. $e_\emptyset = 0$,
- III_G. *whenever $p_1, q_1, \dots, p_m, q_m \in P$ satisfy $p_1^{-1}q_1 \cdots p_m^{-1}q_m = e$ in G , then*

$$v_{p_1}^* v_{q_1} \cdots v_{p_m}^* v_{q_m} = e_{[q_m^{-1}p_m \cdots q_1^{-1}p_1 P]}$$

for all p, q in P and X, Y in \mathcal{J} .

As before, we set $D_s(P) := C^*(\{e_X : X \in \mathcal{J}\}) \subseteq C_s^*(P)$.

By universal property of $C_s^*(P)$ and Lemma 3.1, there exists a homomorphism $\lambda : C_s^*(P) \rightarrow C_r^*(P)$ determined by $\lambda(v_p) = V_p$ and $\lambda(e_X) = E_X$. In particular, $C_s^*(P)$ is non-zero.

It turns out that relation III_G implies the relations I.(ii), II.(i) and II.(iii) from Definition 2.2. Here is an equivalent way of formulating this:

Lemma 3.3. *There is a surjective homomorphism $\pi_s : C^*(P) \rightarrow C_s^*(P)$ sending $C^*(P) \ni v_p$ to $v_p \in C_s^*(P)$ and $C^*(P) \ni e_X$ to $e_X \in C_s^*(P)$.*

Proof. It suffices to check that such a homomorphism exists. We have to show that the relations I.(ii), II.(i) and II.(iii) from Definition 2.2 are satisfied in $C_s^*(P)$. The universal property of $C^*(P)$ will then imply existence of π_s .

II.(i) holds in $C_s^*(P)$ as $e_P \stackrel{\text{III}_G}{=} v_e^* v_e = 1$. To proceed, we first prove a general result about the family of constructible right ideals of P , namely, that it is automatic that \mathcal{J} is closed under finite intersections, i.e.

$$(32) \quad \mathcal{J} = \{q_1^{-1}p_1 \cdots q_m^{-1}p_m P : m \geq 1; p_i, q_i \in P\} \cup \{\emptyset\}.$$

To prove (32), we first show that for every $p_i, q_i \in P$ and every subset X of P ,

$$(33) \quad q_1^{-1}p_1 \cdots q_m^{-1}p_m p_m^{-1}q_m \cdots p_1^{-1}q_1 X = (q_1^{-1}p_1 \cdots q_m^{-1}p_m P) \cap X.$$

We proceed inductively on m :

“ $m = 1$ ”:

$$(34) \quad q_1^{-1}p_1 p_1^{-1}q_1 X = q_1^{-1}((p_1 P) \cap q_1 X) = (q_1^{-1}p_1 P) \cap X.$$

“ $m \rightarrow m + 1$ ”:

$$\begin{aligned} & q_1^{-1}p_1 \cdots q_{m+1}^{-1}p_{m+1} p_{m+1}^{-1}q_{m+1} \cdots p_1^{-1}q_1 X \\ &= (q_1^{-1}p_1 \cdots q_m^{-1}p_m)(q_{m+1}^{-1}p_{m+1} p_{m+1}^{-1}q_{m+1}(p_m^{-1}q_m \cdots p_1^{-1}q_1 X)) \\ (34) \quad &= (q_1^{-1}p_1 \cdots q_m^{-1}p_m)((q_{m+1}^{-1}p_{m+1} P) \cap (p_m^{-1}q_m \cdots p_1^{-1}q_1 X)) \\ &= (q_1^{-1}p_1 \cdots q_{m+1}^{-1}p_{m+1} P) \cap (q_1^{-1}p_1 \cdots q_m^{-1}p_m p_m^{-1}q_m \cdots p_1^{-1}q_1 X) \\ &= (q_1^{-1}p_1 \cdots q_{m+1}^{-1}p_{m+1} P) \cap (q_1^{-1}p_1 \cdots q_m^{-1}p_m P) \cap X \text{ (by induction hypothesis)} \\ &= (q_1^{-1}p_1 \cdots q_{m+1}^{-1}p_{m+1} P) \cap X \text{ (as } q_1^{-1}p_1 \cdots q_{m+1}^{-1}p_{m+1} P \subseteq q_1^{-1}p_1 \cdots q_m^{-1}p_m P). \end{aligned}$$

This proves (33).

We deduce that the right hand side in (32) is closed under finite intersections. This implies by definition of \mathcal{J} that “ \subseteq ” in (32) holds. As “ \supseteq ” obviously holds as well, we have proven (32).

Let us now show that I.(ii) and II.(iii) from Definition 2.2 are satisfied in $C_s^*(P)$. As a special case of (33) ($X = P$), we obtain

$$(35) \quad q_1^{-1}p_1 \cdots q_m^{-1}p_m p_m^{-1}q_m \cdots p_1^{-1}q_1 P = q_1^{-1}p_1 \cdots q_m^{-1}p_m P.$$

Take $p \in P$, $X = q_1^{-1}p_1 \cdots q_m^{-1}p_m P \in \mathcal{J}$ and $Y = s_1^{-1}r_1 \cdots s_n^{-1}r_n P \in \mathcal{J}$. Then

$$\begin{aligned} v_p e_X v_p^* &= v_p e_{[q_1^{-1}p_1 \cdots q_m^{-1}p_m P]} v_p^* \stackrel{(35)}{=} v_p e_{[q_1^{-1}p_1 \cdots q_m^{-1}p_m p_m^{-1}q_m \cdots p_1^{-1}q_1 P]} v_p^* \\ &\stackrel{\text{III}_G}{=} v_p v_{q_1}^* v_{p_1} \cdots v_{q_m}^* v_{p_m} v_{p_m}^* v_{q_m} \cdots v_{p_1}^* v_{q_1} v_p^* \stackrel{\text{III}_G}{=} e_{[p q_1^{-1}p_1 \cdots q_m^{-1}p_m p_m^{-1}q_m \cdots p_1^{-1}q_1 p^{-1}P]} \\ &\stackrel{(35)}{=} e_{[p q_1^{-1}p_1 \cdots q_m^{-1}p_m P]} = e_p X. \end{aligned}$$

This proves I.(ii). Moreover,

$$\begin{aligned} e_X e_Y &= e_{[q_1^{-1}p_1 \cdots q_m^{-1}p_m P]} e_{[s_1^{-1}r_1 \cdots s_n^{-1}r_n P]} \\ &\stackrel{(35)}{=} e_{[q_1^{-1}p_1 \cdots q_m^{-1}p_m p_m^{-1}q_m \cdots p_1^{-1}q_1 P]} e_{[s_1^{-1}r_1 \cdots s_n^{-1}r_n r_n^{-1}s_n \cdots r_1^{-1}s_1 P]} \\ &\stackrel{\text{III}_G}{=} v_{q_1}^* v_{p_1} \cdots v_{q_m}^* v_{p_m} v_{p_m}^* v_{q_m} \cdots v_{p_1}^* v_{q_1} v_{s_1}^* v_{r_1} \cdots v_{s_n}^* v_{r_n} v_{r_n}^* v_{s_n} \cdots v_{r_1}^* v_{s_1} \\ &\stackrel{\text{III}_G}{=} e_{[s_1^{-1}r_1 \cdots s_n^{-1}r_n r_n^{-1}s_n \cdots r_1^{-1}s_1 (q_1^{-1}p_1 \cdots q_m^{-1}p_m p_m^{-1}q_m \cdots p_1^{-1}q_1 P)]} \\ &\stackrel{(33)}{=} e_{[(s_1^{-1}r_1 \cdots s_n^{-1}r_n P) \cap (q_1^{-1}p_1 \cdots q_m^{-1}p_m p_m^{-1}q_m \cdots p_1^{-1}q_1 P)]} \\ &\stackrel{(35)}{=} e_{[(s_1^{-1}r_1 \cdots s_n^{-1}r_n P) \cap (q_1^{-1}p_1 \cdots q_m^{-1}p_m P)]} = e_X \cap Y. \end{aligned}$$

Thus II.(iii) also holds in $C_s^*(P)$. \square

It follows from Corollary 2.10 that $C_s^*(P)$ is generated by the isometries $\{v_p : p \in P\}$. By construction, we have a commutative triangle

$$\begin{array}{ccc} C_s^*(P) & & \\ \pi_s \downarrow & \searrow \lambda & \\ C_s^*(P) & \xrightarrow{\lambda} & C_r^*(P). \end{array}$$

Since $\pi_s(D(P)) = D_s(P)$, we can restrict this triangle to $D(P)$ and obtain another commutative diagram

$$\begin{array}{ccc} D(P) & & \\ \pi_s \downarrow & \searrow \lambda & \\ D_s(P) & \xrightarrow{\lambda} & D_r(P). \end{array}$$

As $\pi_s : D(P) \rightarrow D_s(P)$ is surjective, we deduce from Corollary 2.27

Corollary 3.4. *If the constructible right ideals of P are independent, then $\lambda|_{D_s(P)} : D_s(P) \rightarrow D_r(P)$ is an isomorphism.*

Moreover, we obtain by universal property of $C_s^*(P)$ a homomorphism

$$(36) \quad \Delta : C_s^*(P) \rightarrow C_s^*(P) \otimes_{\max} C_s^*(P), \quad v_p \mapsto v_p \otimes v_p, \quad e_X \mapsto e_X \otimes e_X.$$

In the definition of $C_s^*(P)$, we have used the inclusion $P \subseteq G$. However, the C^* -algebra $C_s^*(P)$ is independent from G (up to canonical isomorphism). Namely, $C_s^*(P)$ can be viewed as $C^*(P)$ with the extra relations III_G by Lemma 3.3. To show independence, let $P \subseteq G_1$ and $P \subseteq G_2$ be two embeddings. We want to see that III_{G_1} and III_{G_2} give the same relations. As we do not add relations if $e_{[q_m^{-1}p_m \cdots q_1^{-1}p_1P]} = 0$ in III_G , all we have to show is that for all $p_1, q_1, \dots, p_m, q_m$ in P ,

$$(37) \quad \begin{aligned} & p_1^{-1}q_1 \cdots p_m^{-1}q_m = e \text{ in } G_1 \text{ and } e_{[q_m^{-1}p_m \cdots q_1^{-1}p_1P]} \neq 0 \\ \Leftrightarrow & p_1^{-1}q_1 \cdots p_m^{-1}q_m = e \text{ in } G_2 \text{ and } e_{[q_m^{-1}p_m \cdots q_1^{-1}p_1P]} \neq 0. \end{aligned}$$

Once this is proven, we conclude that $C_s^*(P)$ is independent from the group into which we embed P . By symmetry, it suffices to prove “ \Rightarrow ”. Take $p_1, q_1, \dots, p_m, q_m$ in P such that $p_1^{-1}q_1 \cdots p_m^{-1}q_m = e$ in G_1 and $e_{[q_m^{-1}p_m \cdots q_1^{-1}p_1P]} \neq 0$. As the latter condition implies $q_m^{-1}p_m \cdots q_1^{-1}p_1P \neq \emptyset$, we can choose $x \in q_m^{-1}p_m \cdots q_1^{-1}p_1P$. Then on $\ell^2(P)$, we have $(V_{p_1}^* V_{q_1} \cdots V_{p_m}^* V_{q_m})(\varepsilon_x) = \varepsilon_x$ as $p_1^{-1}q_1 \cdots p_m^{-1}q_m = e$ in G_1 . But we also have $(V_{p_1}^* V_{q_1} \cdots V_{p_m}^* V_{q_m})(\varepsilon_x) = \varepsilon_{p_1^{-1}q_1 \cdots p_m^{-1}q_m x}$ where this time, the product $p_1^{-1}q_1 \cdots p_m^{-1}q_m x$ is taken in G_2 . Thus we have $p_1^{-1}q_1 \cdots p_m^{-1}q_m x = x$ in G_2 , hence $p_1^{-1}q_1 \cdots p_m^{-1}q_m = e$ in G_2 . This proves (37).

We remark that we can also define $C_s^{*(\cup)}(P)$ (see § 2) and crossed products $A \rtimes_{\alpha, s}^a P$ as in § 2.2. But since these constructions will not be needed, we do not go into the details here.

3.1. Examples of subsemigroups. It is not clear for which semigroups $\pi_s : C^*(P) \rightarrow C_s^*(P)$ is an isomorphism. But in typical examples, we see that condition III_G is already satisfied in $C^*(P)$.

For instance, let (G, P) be a quasi-lattice ordered group as in § 2.4. In that case, III_G is automatically satisfied in $C^*(P)$. Namely, given p, q in P such that $(pP) \cap (qP) \neq \emptyset$, we can find $r \in P$ such that $(pP) \cap (qP) = rP$, and then $v_p^* v_q = v_p^* v_p v_p^* v_q v_q^* v_q = v_p^* v_r v_r^* v_q = v_{p^{-1}r} v_{q^{-1}r}^*$. Applying this several times, we can write $v_{p_1}^* v_{q_1} \cdots v_{p_m}^* v_{q_m}$ as $v_x v_y^*$ for some $x, y \in P$ if $e_{[q_m^{-1}p_m \cdots q_1^{-1}p_1P]} \neq 0$. Now if $p_1^{-1}q_1 \cdots p_m^{-1}q_m = e$ in G , then $xy^{-1} = e$ in G , hence $x = y$. Therefore, $v_{p_1}^* v_{q_1} \cdots v_{p_m}^* v_{q_m} = v_x v_x^*$ is a projection, and we deduce $v_{p_1}^* v_{q_1} \cdots v_{p_m}^* v_{q_m} = (v_{p_1}^* v_{q_1} \cdots v_{p_m}^* v_{q_m})^* (v_{p_1}^* v_{q_1} \cdots v_{p_m}^* v_{q_m}) = e_{[q_m^{-1}p_m \cdots q_1^{-1}p_1P]}$ in $C^*(P)$.

Another class of such examples is given by left Ore semigroups.

Definition 3.5. A semigroup P is called *right reversible* if for every p, q in P , we have $(pP) \cap (qP) \neq \emptyset$.

Definition 3.6. A semigroup is called *left Ore* if it is cancellative (i.e. left and right cancellative) and right reversible.

We have the following

Theorem 3.7 (Ore, Dubreil). *A semigroup P can be embedded into a group G such that $G = P^{-1}P = \{q^{-1}p : p, q \in P\}$ if and only if P is left Ore.*

The reader may consult [Cl-Pr], Theorem 1.24 or [La], § 1.1 for more explanations about this theorem. For later purposes, we also introduce the following

Definition 3.8. *A semigroup P is called left reversible if for every p, q in P , we have $(pP) \cap (qP) \neq \emptyset$.*

Definition 3.9. *A semigroup is called right Ore if it is cancellative and left reversible.*

The analogue of Theorem 3.7 is

Theorem 3.10 (Ore, Dubreil (right version)). *A semigroup P can be embedded into a group G such that $G = PP^{-1} = \{pq^{-1} : p, q \in P\}$ if and only if P is right Ore.*

Now let us see that for a left Ore semigroup, condition III_G is already satisfied in $C^*(P)$. Given p, q in P , there exist by right reversibility r, s in P such that $rp = sq$. Thus $v_q v_p^* = v_s^* v_s v_q v_p^* = v_s^* v_r v_p v_p^*$. Applying this several times, we can write $v_{p_1}^* v_{q_1} \cdots v_{p_m}^* v_{q_m}$ as $v_y^* v_x e_X$ for some $X \in \mathcal{J}$. If $p_1^{-1} q_1 \cdots p_m^{-1} q_m = e$ holds in $G = P^{-1}P$, then $y^{-1}x = e$ in G , hence $x = y$. Thus we again conclude that $v_{p_1}^* v_{q_1} \cdots v_{p_m}^* v_{q_m} = v_x^* v_x e_X$ is a projection, and the same argument as in the quasi-lattice ordered case gives $v_{p_1}^* v_{q_1} \cdots v_{p_m}^* v_{q_m} = e_{[q_m^{-1} p_m \cdots q_1^{-1} p_1 P]}$ in $C^*(P)$.

3.2. Conditional expectations. We conclude this section with a few observations which will be used later on. First of all, there is a faithful conditional expectation $\mathcal{E}_r : \mathcal{L}(\ell^2(P)) \rightarrow \ell^\infty(P) \subseteq \mathcal{L}(\ell^2(P))$ characterized by

$$\langle \mathcal{E}_r(T) \varepsilon_x, \varepsilon_x \rangle = \langle T \varepsilon_x, \varepsilon_x \rangle \text{ for all } T \in \mathcal{L}(\ell^2(P)), x \in P.$$

Here $\ell^\infty(P)$ acts on $\ell^2(P)$ by multiplication operators.

Lemma 3.11. *If P embeds into a group G , then $\mathcal{E}_r(C_r^*(P)) = D_r(P)$.*

Proof. As $D_r(P) \subseteq \ell^\infty(P)$, it is clear that $\mathcal{E}_r(C_r^*(P))$ contains $D_r(P)$. It remains to prove “ \subseteq ”. By the definition of the reduced semigroup C*-algebra, we have

$$C_r^*(P) = \overline{\text{span}}(\{V_{p_1}^* V_{q_1} \cdots V_{p_m}^* V_{q_m} : m \in \mathbb{Z}_{>0}; p_i, q_i \in P \text{ for all } 1 \leq i \leq m\}).$$

So it suffices to prove that for every $p_1, q_1, \dots, p_m, q_m \in P$, $\mathcal{E}_r(V_{p_1}^* V_{q_1} \cdots V_{p_m}^* V_{q_m}) \in D_r(P)$. Set $V := V_{p_1}^* V_{q_1} \cdots V_{p_m}^* V_{q_m}$. It is clear that for every $x \in P$, $V \varepsilon_x$ is either 0 or of the form ε_y for some $y \in P$. Now assume that $\mathcal{E}_r(V) \neq 0$. Then there must be $x \in P$ with $V \varepsilon_x = \varepsilon_x$. But this implies that $p_1^{-1} q_1 \cdots p_m^{-1} q_m x = x$, and thus $p_1^{-1} q_1 \cdots p_m^{-1} q_m = e$ in G . Lemma 3.1 implies that $V = E_{[q_m^{-1} p_m \cdots q_1^{-1} p_1 P]}$ lies in $D_r(P)$. \square

Remark 3.12. This lemma implies that $D_r(P) = C_r^*(P) \cap \ell^\infty(P)$ if P embeds into a group. At this point, we see that it is convenient to work with the family \mathcal{J} which is closed under pre-images (with respect to left multiplication), see Remark 2.3.

Now let P be a subsemigroup of a group G , and let ${}^*\text{-alg}(P)$ be the sub- * -algebra of $C_s^*(P)$ generated by the v_p , $p \in P$. Set for $g \in G$

$$(38) \quad D_g := \text{span}(\{v_{p_1}^* v_{q_1} \cdots v_{p_m}^* v_{q_m} : m \geq 1; p_i, q_i \in P \text{ and } p_1^{-1} q_1 \cdots p_m^{-1} q_m = g\})$$

as a subspace of ${}^*\text{-alg}(P)$. We then obviously have ${}^*\text{-alg}(P) = \sum_{g \in G} D_g$.

Lemma 3.13. *Assume that P embeds into a group G and that the constructible right ideals of P are independent. Then there is a conditional expectation $\mathcal{E}_s : C_s^*(P) \rightarrow D_s(P)$ with*

$$(39) \quad \mathcal{E}_s|_{D_g} = 0 \text{ if } g \neq e \text{ and } \mathcal{E}_s|_{D_e} = \text{id}_{D_e};$$

$$(40) \quad \ker(\lambda) \cap C_s^*(P)_+ = \ker(\mathcal{E}_s) \cap C_s^*(P)_+,$$

where $C_s^*(P)_+$ denotes the set of positive elements in $C_s^*(P)$.

Proof. Since we assume that the constructible right ideals of P are independent, we know that $\lambda|_{D_s(P)}$ is an isomorphism. Thus we can set

$$\mathcal{E}_s := (\lambda|_{D_s(P)})^{-1} \circ \mathcal{E}_r \circ \lambda : C_s^*(P) \rightarrow D_s(P).$$

We have

$$\mathcal{E}_r(V_{p_1}^* V_{q_1} \cdots V_{p_m}^* V_{q_m}) = \begin{cases} E_{[q_m^{-1} p_m \cdots q_1^{-1} p_1 P]} & \text{if } p_1^{-1} q_1 \cdots p_m^{-1} q_m = e, \\ 0 & \text{if } p_1^{-1} q_1 \cdots p_m^{-1} q_m \neq e. \end{cases}$$

Therefore we obviously have $\mathcal{E}_s|_{D_g} = 0$ if $g \neq e$. And for $p_1, q_1, \dots, p_m, q_m \in P$ with $p_1^{-1} q_1 \cdots p_m^{-1} q_m = e$ in G , we have

$$\begin{aligned} \mathcal{E}_s(v_{p_1}^* v_{q_1} \cdots v_{p_m}^* v_{q_m}) &= ((\lambda|_{D_s(P)})^{-1} \circ \mathcal{E}_r)(V_{p_1}^* V_{q_1} \cdots V_{p_m}^* V_{q_m}) \\ &= (\lambda|_{D_s(P)})^{-1}(E_{[q_m^{-1} p_m \cdots q_1^{-1} p_1 P]}) = e_{[q_m^{-1} p_m \cdots q_1^{-1} p_1 P]} \stackrel{\text{III}_G}{=} v_{p_1}^* v_{q_1} \cdots v_{p_m}^* v_{q_m}. \end{aligned}$$

□

4. AMENABILITY

In this section, our goal is to study the relationship between semigroups and their semigroup C^* -algebras in the context of amenability. It turns out that, using our constructions of semigroup C^* -algebras, there are strong parallels between the semigroup case and the group case. Indeed, one of our main goals in this section is to show that the analogues of [Br-Oz], Chapter 2, Theorem 6.8 (1)–(7) are also equivalent in the case of semigroups (under certain assumptions on the semigroups). Apart from this result, we also prove a few additional statements.

Let us first state our main result. To do so, we recall some definitions. The reader may find more explanations in [Pa].

Definition 4.1. *A discrete semigroup P is left amenable if there exists a left invariant mean on $\ell^\infty(P)$, i.e. a state μ on $\ell^\infty(P)$ such that for every $p \in P$ and $f \in \ell^\infty(P)$, $\mu(f(p\sqcup)) = \mu(f)$. Here $f(p\sqcup)$ is the composition of f after left multiplication with p .*

Definition 4.2. An approximate left invariant mean on a discrete semigroup P is a net $(\mu_i)_i$ in $\ell^1(P)$ of positive elements of norm 1 with the property that

$$\lim_i \|\mu_i - \mu_i(p\sqcup)\|_{\ell^1(P)} = 0 \text{ for all } p \in P.$$

Here $\mu_i(p\sqcup)$ again is the composition of μ_i after left multiplication with p .

Definition 4.3. A discrete semigroup P satisfies the strong Følner condition if for every finite subset $C \subseteq P$ and every $\varepsilon > 0$, there exists a non-empty finite subset $F \subseteq P$ such that $|(pF)\Delta F|/|F| < \varepsilon$ for all $p \in C$.

Here Δ stands for symmetric difference.

4.1. Statements. Let P be a discrete left cancellative semigroup. We consider the following statements:

- 1) P is left amenable.
- 2) P has an approximate left invariant mean.
- 3) P satisfies the strong Følner condition.
- 4) There exists a net $(\xi_i)_i$ in $\ell^2(P)$ with $\|\xi_i\| = 1$ for all i and $\lim_i \|V_p \xi_i - \xi_i\| = 0$ for all $p \in P$.
- 5) There exists a net $(\xi_i)_i$ in $C_c(P) \subseteq \ell^2(P)$ with $\|\xi_i\| = 1$ for all i such that $\lim_i \langle V_{p_1}^* V_{q_1} \cdots V_{p_n}^* V_{q_n} \xi_i, \xi_i \rangle = 1$ for all $n \in \mathbb{Z}_{>0}$; $p_1, q_1, \dots, p_n, q_n \in P$.
- 6) The left regular representation $\lambda : C_s^*(P) \rightarrow C_r^*(P)$ is an isomorphism and there exists a non-zero character on $C_s^*(P)$.
- 7) There exists a non-zero character on $C_r^*(P)$.

Our goal is to show that for a discrete left cancellative semigroup, we always have “1) \Leftrightarrow 2) \Leftrightarrow 3) \Rightarrow 4) \Rightarrow 5)” and “6) \Rightarrow 7) \Rightarrow 1)”, and that if P is also right cancellative and if the constructible right ideals are independent (see Definition 2.26), then “5) \Rightarrow 6)” holds as well. With Corollary 2.27 in mind, it is not surprising that independence of the family of constructible right ideals plays a role in the context of amenability. Moreover, note that 6) only makes sense if P can be embedded into a group. Thus our assumption that P should be cancellative is certainly necessary, and as a part of “5) \Rightarrow 6)”, we will prove that 5) implies that P embeds into a group. In addition, we will see in Remark 4.11 that 5) implies 7) for every discrete left cancellative semigroup.

Before we start with the proofs, let us remark that the equivalence of 1), 2) and 3) for discrete left cancellative semigroups is certainly known, and that these equivalences can be proven as in the group case. We include proofs of these equivalences for the sake of completeness. Moreover, the implications “3) \Rightarrow 4) \Rightarrow 5)” and “6) \Rightarrow 7)” are easy. And for the implication “7) \Rightarrow 1)”, the proof in the group case as presented in [Br-Oz], Chapter 2, Theorem 6.8 carries over to the case of semigroups. Again, for the sake of completeness, we present a proof for this implication. Both for the equivalence of 1), 2) and 3) as well as for the implication “7) \Rightarrow 1)”, we only have to check that in the proofs of the corresponding statements in the group case, we can avoid taking inverses as this is in general not possible in semigroups. And finally, to

prove “5) \Rightarrow 6)” under the additional assumptions that P is right cancellative and that the constructible right ideals of P are independent, we adapt A. Nica’s ideas in [Ni], § 4.4 to our situation.

4.2. Proofs. We start with “1) \Leftrightarrow 2)”. First assume that there is a left invariant mean μ on $\ell^\infty(P)$. As the unit ball of $\ell^1(P)$ is weak*-dense in the unit ball of $\ell^1(P)'' \cong \ell^\infty(P)'$, there exists a net $(\mu_i)_i$ of positive elements in $\ell^1(P)$ with norm 1 which converges to μ in the weak*-topology. This means that $\lim_i \mu_i(f) = \mu(f)$ for every $f \in \ell^\infty(P)$. We want to show that for every $p \in P$ and $f \in \ell^\infty(P)$, $\lim_i \mu_i(f) - (\mu_i(p\sqcup))(f) = 0$. To prove this, take $f \in \ell^\infty(P)$, $p \in P$ and define a function $g \in \ell^\infty(P)$ by $g(q) := \begin{cases} f(r) & \text{if } q = pr \\ 0 & \text{else.} \end{cases}$ Then $\lim_i (\mu_i(g(p\sqcup)) - \mu_i(g)) = \mu(g(p\sqcup)) - \mu(g) = 0$ as μ is left invariant. At the same time,

$$\begin{aligned} \mu_i(g(p\sqcup)) - \mu_i(g) &= \sum_q \mu_i(q)g(pq) - \sum_q \mu_i(q)g(q) \\ &= \sum_q \mu_i(q)g(pq) - \sum_q \mu_i(pq)g(pq) - \sum_{q \notin pP} \mu_i(q) \underbrace{g(q)}_{=0} \\ &= \sum_q \mu_i(q)f(q) - \sum_q \mu_i(pq)f(q) = \mu_i(f) - (\mu_i(p\sqcup))(f). \end{aligned}$$

This shows that we indeed have $\lim_i \mu_i(f) - (\mu_i(p\sqcup))(f) = 0$. Hence, for every $n \in \mathbb{Z}_{>0}$ and $p_1, \dots, p_n \in P$, $(0, \dots, 0)$ lies in the weak closure of

$$(41) \quad \{(\nu - \nu(p_j\sqcup))_{j=1, \dots, n} : \nu \in \ell^1(P), \nu \geq 0, \|\nu\| \leq 1\}.$$

As this set is convex, it follows from the Hahn-Banach separation theorem that its weak and norm closures coincide. That $(0, \dots, 0)$ lies in the norm closure of (41) tells us that P has an approximate left invariant mean. This proves “1) \Rightarrow 2)”.

For the reverse implication, assume that P has an approximate left invariant mean $(\mu_i)_i$. By definition, this means

$$(42) \quad \lim_i \|\mu_i - \mu_i(p\sqcup)\|_{\ell^1(P)} = 0 \text{ for all } p \in P.$$

Moreover, we have $\|\mu_i - \mu_i(p\sqcup)\|_{\ell^1(P)} \geq \|\mu_i\|_{\ell^1(P)} - \|\mu_i(p\sqcup)\|_{\ell^1(P)} = \sum_{q \notin pP} |\mu_i(q)|$. It follows that

$$(43) \quad \lim_i \sum_{q \notin pP} |\mu_i(q)| = 0.$$

Now $\ell^\infty(P)' \cong \ell^1(P)''$, and by the theorem of Banach-Alaoglu, the unit ball of $\ell^1(P)''$ is weak*-compact. Hence by passing to a suitable subnet if necessary, we may assume that the net $(\mu_i)_i$ converges to an element $\mu \in \ell^1(P)'' \cong \ell^\infty(P)'$ in the weak*-topology. μ has to be a state on $\ell^\infty(P)$ as the μ_i are positive with norm 1.

For every $f \in \ell^\infty(P)$ and $p \in P$ we have

$$\begin{aligned}
& |\mu(f(p\sqcup)) - \mu(f)| = \lim_i |\mu_i(f(p\sqcup)) - \mu_i(f)| \\
&= \lim_i \left| \sum_{q \in P} \mu_i(q) f(pq) - \sum_{q \in P} \mu_i(q) f(q) \right| \\
&= \lim_i \left| \sum_{q \in P} (\mu_i(q) - \mu_i(pq)) f(pq) - \sum_{q \notin pP} \mu_i(q) f(q) \right| \\
&\leq \lim_i \left(\|\mu_i - \mu_i(p\sqcup)\|_{\ell^1(P)} \cdot \|f\|_{\ell^\infty(P)} + \sum_{q \notin pP} |\mu_i(q)| \|f\|_{\ell^\infty(P)} \right) \\
&= 0
\end{aligned}$$

by (42) and (43). Thus μ is a left invariant mean. This proves “2) \Rightarrow 1)”.

Let us prove “**1**) \Leftrightarrow **3**)”. First of all, if P has an approximate left invariant mean $(\mu_i)_i$, then we always have

$$(44) \quad \lim_i \|\mu_i(p^{-1}\sqcup) - \mu_i\|_{\ell^1(P)} = 0,$$

where $\mu_i(p^{-1}\sqcup)(q) = \begin{cases} \mu_i(q') & \text{if } q = pq' \text{ for some } q' \in P \\ 0 & \text{if } q \notin pP \end{cases}$. The reason is that we have

$$\begin{aligned}
& \|\mu_i(p^{-1}\sqcup) - \mu_i\|_{\ell^1(P)} = \sum_{q \in pP} |\mu_i(p^{-1}\sqcup)(q) - \mu_i(q)| + \sum_{q \notin pP} |\mu_i(q)| \\
&= \sum_{q' \in P} |\mu_i(q') - \mu_i(pq')| + \sum_{q \notin pP} |\mu_i(q)| = \|\mu_i - \mu_i(p\sqcup)\|_{\ell^1(P)} + \sum_{q \notin pP} |\mu_i(q)|
\end{aligned}$$

and $\lim_i \sum_{q \notin pP} |\mu_i(q)| = 0$ by (43).

Now, assume that P has an approximate left invariant mean. Let C be a finite subset P and let $\varepsilon > 0$ be given. By 2) and the fact proven above that every approximate left invariant mean $(\mu_i)_i$ satisfies (44), there exists a positive ℓ^1 -function μ of ℓ^1 -norm 1 with

$$(45) \quad \sum_{p \in C} \|\mu(p^{-1}\sqcup) - \mu\|_{\ell^1(P)} < \varepsilon.$$

For $t \in [0, 1]$, we set $F(\mu, t) := \{q \in P: \mu(q) > t\}$. We claim that for a suitable choice of t , the inequality $\max_{p \in C} |pF(\mu, t)\Delta F(\mu, t)|/|F(\mu, t)| < \varepsilon$ holds. We have

$$\begin{aligned}
& \|\mu(p^{-1}\sqcup) - \mu\|_{\ell^1(P)} = \sum_{q \in P} |(\mu(p^{-1}\sqcup) - \mu)(q)| \\
&= \sum_{q \in P} \int_0^1 |\mathbb{1}_{[0, \mu(p^{-1}\sqcup)(q)]}(t) - \mathbb{1}_{[0, \mu(q)]}(t)| dt \\
&= \sum_{q \in P} \int_0^1 |\mathbb{1}_{F(\mu(p^{-1}\sqcup), t)}(q) - \mathbb{1}_{F(\mu, t)}(q)| dt = \int_0^1 |pF(\mu, t)\Delta F(\mu, t)| dt
\end{aligned}$$

and

$$\begin{aligned} \int_0^1 \varepsilon |F(\mu, t)| dt &= \varepsilon \int_0^1 \sum_{q \in P} \mathbb{1}_{F(\mu, t)}(q) dt = \varepsilon \sum_{q \in P} \int_0^1 \mathbb{1}_{F(\mu, t)}(q) dt \\ &= \varepsilon \sum_{q \in P} \int_0^1 \mathbb{1}_{[0, \mu(q)]}(t) dt = \varepsilon \sum_{q \in P} \mu(q) = \varepsilon. \end{aligned}$$

Plugging these two inequalities into (45), we obtain

$$\int_0^1 \varepsilon |F(\mu, t)| dt > \int_0^1 \sum_{p \in C} |(pF(\mu, t)) \Delta F(\mu, t)| dt$$

Thus there is $t \in [0, 1]$ with $\varepsilon |F(\mu, t)| > \sum_{p \in C} |(pF(\mu, t)) \Delta F(\mu, t)|$. Therefore P satisfies the strong Følner condition. So we have proven “2) \Rightarrow 3)”.

To prove the reverse implication, observe that 3) tells us that there exists a net $(F_i)_i$ of non-empty finite subsets of P such that $\lim_i |(pF_i) \Delta F_i| / |F_i| = 0$ for all $p \in P$. Set $\mu_i := \frac{1}{|F_i|} \mathbb{1}_{F_i}$. It is clear that $(\mu_i)_i$ is a net of positive ℓ^1 -functions of ℓ^1 -norm 1. Moreover, $\|\mu_i - \mu_i(p \sqcup)\|_{\ell^1(P)} \leq \|\mu_i(p^{-1} \sqcup) - \mu_i\|_{\ell^1(P)} = \|\frac{1}{|F_i|} (\mathbb{1}_{pF_i} - \mathbb{1}_{F_i})\|_{\ell^1(P)} = |(pF_i) \Delta F_i| / |F_i| \rightarrow_i 0$ for all p in P . Thus $(\mu_i)_i$ is an approximate left invariant mean. This proves “3) \Rightarrow 2)”.

To prove “3) \Rightarrow 4)”, first note that since P satisfies the strong Følner condition, there is a net $(F_i)_i$ of non-empty finite subsets of P with $\lim_i |(pF_i) \Delta F_i| / |F_i| = 0$ for all $p \in P$. Now set $\xi_i := |F_i|^{-\frac{1}{2}} \mathbb{1}_{F_i}$. Here $\mathbb{1}_{F_i}$ is the characteristic function of $F_i \subseteq P$. It is clear that every ξ_i lies in $\ell^2(P)$ and has norm 1. Moreover, for every $p \in P$, $V_p \xi_i - \xi_i = |F_i|^{-\frac{1}{2}} (\mathbb{1}_{pF_i} - \mathbb{1}_{F_i})$. It follows that $\|V_p \xi_i - \xi_i\|^2 = |(pF_i) \Delta F_i| / |F_i| \rightarrow_i 0$ for all $p \in P$. This proves “3) \Rightarrow 4)”.

“4) \Rightarrow 5)”: By an approximation argument, we can without loss of generality assume that the ξ_i from 4) all lie in $C_c(P)$. We have by 4) that $\lim_i \|V_p \xi_i - \xi_i\| = 0$ for all $p \in P$ and also $\|V_p^* \xi_i - \xi_i\| \leq \|V_p^*\| \cdot \|\xi_i - V_p \xi_i\| \rightarrow_i 0$ for all $p \in P$. Hence

$$\begin{aligned} & |\langle V_{p_1}^* V_{q_1} \cdots V_{p_n}^* V_{q_n} \xi_i, \xi_i \rangle - 1| \\ &= \left| \sum_{j=1}^n \left(\langle V_{p_1}^* V_{q_1} \cdots V_{p_j}^* V_{q_j} \xi_i, \xi_i \rangle - \langle V_{p_1}^* V_{q_1} \cdots V_{p_{j-1}}^* V_{q_{j-1}} V_{p_j}^* \xi_i, \xi_i \rangle \right. \right. \\ &\quad \left. \left. + \langle V_{p_1}^* V_{q_1} \cdots V_{p_{j-1}}^* V_{q_{j-1}} V_{p_j}^* \xi_i, \xi_i \rangle - \langle V_{p_1}^* V_{q_1} \cdots V_{p_{j-1}}^* V_{q_{j-1}} \xi_i, \xi_i \rangle \right) \right| \\ &\leq \sum_{j=1}^n \|V_{q_j} \xi_i - \xi_i\| + \|V_{p_j}^* \xi_i - \xi_i\| \rightarrow_i 0 \end{aligned}$$

for all $n \in \mathbb{Z}_{>0}$ and $p_1, q_1, \dots, p_n, q_n \in P$. This proves “4) \Rightarrow 5)”.

“6) \Rightarrow 7)” is trivial.

For “7) \Rightarrow 1)”, let $\chi : C_r^*(P) \rightarrow \mathbb{C}$ be a non-zero character. Viewing χ as a state, we can extend it by the theorem of Hahn-Banach to a state on $\mathcal{L}(\ell^2(P))$. We then restrict the extension to $\ell^\infty(P) \subseteq \mathcal{L}(\ell^2(P))$ and call this restriction μ . The point is that by construction, $\mu|_{C_r^*(P)} = \chi$ is multiplicative, hence $C_r^*(P)$ is in the multiplicative domain of μ . Thus we obtain for every $f \in \ell^\infty(P)$ and $p \in P$

$$\mu(f(p\sqcup)) = \mu(V_p^* f V_p) = \mu(V_p^*)\mu(f)\mu(V_p) = \mu(V_p)^*\mu(V_p)\mu(f) = \mu(f).$$

Thus μ is a left invariant mean on $\ell^\infty(P)$. Hence we have proven “7) \Rightarrow 1)”.

It remains to discuss the implication “5) \Rightarrow 6)”. We start with the following

Lemma 4.4. *5) implies that P is left reversible.*

Proof. Let $(\xi_i)_i$ be a net as in 5). For $p_1, p_2 \in P$, we have $\lim_i \langle V_{p_1} V_{p_1}^* V_{p_2} V_{p_2}^* \xi_i, \xi_i \rangle = 1$. In particular, $V_{p_1} V_{p_1}^* V_{p_2} V_{p_2}^* \neq 0$. But $V_{p_1} V_{p_1}^* V_{p_2} V_{p_2}^* = E_{[(p_1 P) \cap (p_2 P)]}$, hence $(p_1 P) \cap (p_2 P) \neq \emptyset$. This shows that P is left reversible. \square

Corollary 4.5. *If P is cancellative and 5) holds, then P embeds into a group G such that $G = PP^{-1}$.*

Proof. This follows from the previous lemma and Theorem 3.10. \square

Lemma 4.6. *A subsemigroup P of a group is left reversible if and only if there exists a non-zero character on $C_s^*(P)$.*

Proof. If χ is a non-zero character on $C_s^*(P)$, then for every $p_1, p_2 \in P$, we have $\chi(e_{[(p_1 P) \cap (p_2 P)]}) = \chi(v_{p_1} v_{p_1}^* v_{p_2} v_{p_2}^*) = \chi(v_{p_1})\chi(v_{p_1}^*)\chi(v_{p_2})\chi(v_{p_2}^*) = 1$. This implies that $(p_1 P) \cap (p_2 P) \neq \emptyset$ because otherwise $e_{[(p_1 P) \cap (p_2 P)]}$ would vanish.

If P is left reversible, then by universal property of $C_s^*(P)$, there is a homomorphism $C_s^*(P) \rightarrow \mathbb{C}$ sending $C_s^*(P) \ni v_p$ to $1 \in \mathbb{C}$ and $C_s^*(P) \ni e_X$ to $1 \in \mathbb{C}$ if $X \neq \emptyset$ and to $0 \in \mathbb{C}$ if $X = \emptyset$ for every $p \in P$ and $X \in \mathcal{J}$. This is compatible with relation III_G as $q_m^{-1} p_m \cdots q_1^{-1} p_1 P$ is never empty. The last fact follows inductively on m using the observation that for every non-empty right ideal X of P , we have $q^{-1} p X = q^{-1}((pX) \cap (qP))$, and that $(pX) \cap (qP) \neq \emptyset$ by left reversibility. \square

It remains to prove that 5) implies that $\lambda : C_s^*(P) \rightarrow C_r^*(P)$ is an isomorphism if P is cancellative (not only left cancellative, but also right cancellative) and if the constructible right ideals of P are independent. Recall the definition of D_g from (38). For a positive functional φ on $C_s^*(P)$, we define the d-support of φ as $\text{d-supp}(\varphi) := \{g \in G : \varphi|_{D_g} \neq 0\}$. Moreover, we set

$$\mathcal{V} := \{v_{p_1}^* v_{q_1} \cdots v_{p_n}^* v_{q_n} : n \in \mathbb{Z}_{>0}; p_i, q_i \in P\}.$$

Our aim is to show

Theorem 4.7. *Let P be a subsemigroup of a group G , and assume that the constructible right ideals of P are independent. If there exists a net $(\varphi_i)_i$ of states on $C_s^*(P)$ with finite d -support such that $\lim_i \varphi_i(v) = 1$ for every $0 \neq v$ in \mathcal{V} , then $\lambda : C_s^*(P) \rightarrow C_r^*(P)$ is an isomorphism.*

Note that this is the analogue of the implication “(5) \Rightarrow (6)” in [Br-Oz], Chapter 2, Theorem 6.8 in the group case. To prove the theorem, we first show

Lemma 4.8. *Let φ be a positive functional on $C_s^*(P)$ with finite d -support. We then have for all $x \in C_s^*(P)$:*

$$(46) \quad |\varphi(x)|^2 \leq |d\text{-supp}(\varphi)| \|\varphi\| \varphi(\mathcal{E}_s(x^*x)).$$

Here \mathcal{E}_s is the conditional expectation from Lemma 3.13.

Proof. It certainly suffices to prove our assertion for x in $^*\text{-alg}(P) = \sum_{g \in G} D_g$. Take such an element x . Let $d\text{-supp}(\varphi) = \{g_1, \dots, g_n\}$. We can find a finite subset $F \subseteq G$ so that $x = \sum_{g \in F} x_g$ with $x_g \in D_g$ and $d\text{-supp}(\varphi) \subseteq F$, i.e. $\{g_1, \dots, g_n\} \subseteq F$. Then $\varphi(x) = \sum_{g \in F} \varphi(x_g) = \sum_{j=1}^n \varphi(x_{g_j})$. Thus, using the Cauchy-Schwarz inequality twice, we obtain

$$\begin{aligned} |\varphi(x)|^2 &= \left| \sum_{j=1}^n \varphi(x_{g_j}) \right|^2 = |\langle (\varphi(x_{g_j}))_j, (1)_j \rangle_{\mathbb{C}^n}|^2 \leq \|(\varphi(x_{g_j}))_j\|_{\mathbb{C}^n}^2 \|(1)_j\|_{\mathbb{C}^n}^2 \\ &= n \sum_{j=1}^n |\varphi(x_{g_j})|^2 = n \sum_{j=1}^n |\langle x_{g_j}, 1 \rangle_{\varphi}|^2 \leq n \|\varphi\| \sum_{j=1}^n \varphi(x_{g_j}^* x_{g_j}). \end{aligned}$$

Hence it suffices to prove $\sum_{j=1}^n x_{g_j}^* x_{g_j} \leq \mathcal{E}_s(x^*x)$. We have by (39) and because of $D_g^* D_h \subseteq D_{g^{-1}h}$ for all $g, h \in G$ that

$$\mathcal{E}_s(x^*x) = \sum_{g, h \in F} \mathcal{E}_s(x_g^* x_h) = \sum_{g, h \in F} \delta_{g, h} x_g^* x_h = \sum_{g \in F} x_g^* x_g \geq \sum_{j=1}^n x_{g_j}^* x_{g_j}.$$

This proves (4.8). \square

Proposition 4.9. *$\lambda : C_s^*(P) \rightarrow C_r^*(P)$ is an isomorphism if the set of positive functionals on $C_s^*(P)$ with finite d -support is dense in the space of all positive functionals on $C_s^*(P)$ in the weak*-topology.*

Proof. Take $x \in \ker(\lambda)$. Passing over to x^*x if necessary, we may assume $x \geq 0$. Take a positive functional φ on $C_s^*(P)$ with finite d -support. We then have because of $\lambda(x) = 0$ that $\lambda(x^*x) = 0$, thus $\mathcal{E}_s(x^*x) = 0$ by (40). Hence it follows from (4.8) that $\varphi(x) = 0$. So we have shown that $\varphi(x) = 0$ for every positive functional on $C_s^*(P)$ with finite d -support. By our assumption in the proposition, the positive functionals with finite d -support are weak*-dense in the space of all positive functionals. Hence $\varphi(x) = 0$ for every positive functional φ on $C_s^*(P)$. This however implies that $x = 0$. We conclude that λ must be injective, hence an isomorphism. \square

Actually, the converse of the proposition is valid as well, and is simpler to prove. To proceed, we need another

Lemma 4.10. *Let φ and ϕ be positive functionals on $C_s^*(P)$. Then there exists a unique positive functional ψ on $C_s^*(P)$ such that $\psi(v) = \varphi(v)\phi(v)$ for all $v \in \mathcal{V}$.*

Proof. Just set $\psi = (\varphi \otimes \phi) \circ \Delta$ with Δ given by (36). \square

Finally, with all these preparations, we can prove our theorem.

Proof of Theorem 4.7. Let ϕ be a positive functional on $C_s^*(P)$. Let φ_i be the states given by the hypothesis of our theorem, they satisfy

$$(47) \quad \lim_i \varphi_i(v) = 1 \text{ for every } 0 \neq v \in \mathcal{V}.$$

By Lemma 4.10, there exists a net $(\phi_i)_i$ of positive functionals on $C_s^*(P)$ such that for all i ,

$$(48) \quad \phi_i(v) = \varphi_i(v)\phi(v) \text{ for all } v \in \mathcal{V}.$$

In particular, $\|\phi_i\| = \|\phi\|$ since $\phi_i(1) = \phi(1) = \|\phi\|$. It is then clear that for every i , $\text{d-supp}(\phi_i) \subseteq \text{d-supp}(\varphi_i)$ is finite. Moreover, we have $\lim_i \phi_i(v) = \phi(v)$ for all $v \in \mathcal{V}$. This is clear if $v = 0$, and if $v \neq 0$ it follows from (48) and (47). Thus $\lim_i \phi_i(x) = \phi(x)$ for all $x \in {}^*\text{-alg}(P)$, and since $\|\phi_i\| = \|\phi\|$ for all i , we conclude that we actually have $\lim_i \phi_i(x) = \phi(x)$ for all $x \in C_s^*(P)$. In other words, the net $(\phi_i)_i$ converges to ϕ in the weak*-topology. Thus we have seen that the positive functionals with finite d-support are weak*-dense in the space of all positive functionals. By Proposition 4.9, this implies that $\lambda : C_s^*(P) \rightarrow C_r^*(P)$ is an isomorphism. This completes the proof of our theorem. \square

“5) \Rightarrow 6)” if P is cancellative and if the constructible right ideals of P are independent: Assume that P is cancellative and that the constructible right ideals of P are independent. We have already seen that 5) implies that P is left reversible in Lemma 4.4. Thus P embeds into a group by Corollary 4.5, and there is a non-zero character on $C_s^*(P)$ by Lemma 4.6. It remains to prove that $\lambda : C_s^*(P) \rightarrow C_r^*(P)$ is an isomorphism. By Theorem 4.7, it suffices to prove that there exists a net $(\varphi_i)_i$ of states on $C_s^*(P)$ with finite d-support such that $\lim_i \varphi_i(v) = 1$ for every $0 \neq v \in \mathcal{V}$.

Now take the net $(\xi_i)_i$ in $C_c(P)$ from 5), and set for all i : $\varphi_i(x) := \langle \lambda(x)\xi_i, \xi_i \rangle$ for every $x \in C_s^*(P)$. It is clear that these φ_i are states and that we have $\lim_i \varphi_i(v) = 1$ for every $0 \neq v \in \mathcal{V}$. Moreover, for every i , set $\text{supp}(\xi_i) := \{p \in P : \xi_i(p) \neq 0\}$. By assumption (see 5)), $\text{supp}(\xi_i)$ is a finite set for every i . We have $\varphi_i(v_{p_1}^* v_{q_1} \cdots v_{p_n}^* v_{q_n}) = \langle V_{p_1}^* V_{q_1} \cdots V_{p_n}^* V_{q_n} \xi_i, \xi_i \rangle \neq 0$ only if there are x, y in $\text{supp}(\xi_i)$ with $p_1^{-1} q_1 \cdots p_n^{-1} q_n x = y$. But this implies $p_1^{-1} q_1 \cdots p_n^{-1} q_n \in (\text{supp}(\xi_i))(\text{supp}(\xi_i))^{-1}$, or in other words, that $\text{d-supp}(\varphi_i) \subseteq (\text{supp}(\xi_i))(\text{supp}(\xi_i))^{-1}$. As $\text{supp}(\xi_i)$ is a finite set for every i , this proves that for every i , φ_i has finite d-support. This shows that the conditions in Theorem 4.7 are satisfied, hence that $\lambda : C_s^*(P) \rightarrow C_r^*(P)$ is an isomorphism. Thus

we have seen that 5) implies 6) if P is cancellative and if the constructible right ideals of P are independent.

Remark 4.11. We point out that 5) implies 7) for every discrete left cancellative semigroup P . Just set χ as the weak*-limit of the vector states $\langle \sqcup \xi_i, \xi_i \rangle$ of $C_r^*(P)$ where the ξ_i are provided by 5). It is easy to see that χ is multiplicative.

4.3. Additional results. There are a few related statements we now turn to. First of all, we can of course consider the following

Definition 4.12. A discrete semigroup P is called right amenable if there exists a right invariant mean on $\ell^\infty(P)$.

A right amenable semigroup P is always right reversible, i.e. for every $p_1, p_2 \in P$, we have $(Pp_1) \cap (Pp_2) \neq \emptyset$. This is the analogue of [Pa], Proposition (1.23) if we replace “left” in [Pa] by “right”. If P is cancellative and right reversible, then P embeds into a group G such that $G = P^{-1}P$ (see Theorem 3.10). G is amenable if P is right amenable (this is the right version of [Pa], Proposition (1.27)).

Proposition 4.13. Let P be a cancellative, right amenable semigroup. Then $\lambda^{(\cup)} : C^{*(\cup)}(P) \rightarrow C_r^*(P)$ is an isomorphism.

Proof. Consider the embedding $P \hookrightarrow G = P^{-1}P$ from above. We know that $C^{*(\cup)}(P) \cong D^{(\cup)}(P) \rtimes_{\tau^{(\cup)}}^e P$ by Lemma 2.14. By dilation theory for semigroup crossed products by endomorphisms (see [La]), there exists a C^* -algebra D_∞ with an embedding $D^{(\cup)}(P) \xrightarrow{i} D_\infty$ and an action τ_∞ of G on D_∞ whose restriction to P leaves $D^{(\cup)}(P)$ invariant and coincides with $\tau^{(\cup)}$. Moreover, $D^{(\cup)}(P) \rtimes_{\tau^{(\cup)}}^e P$ embeds into $D_\infty \rtimes_{\tau_\infty} G$. Let us denote this embedding $D^{(\cup)}(P) \rtimes_{\tau^{(\cup)}}^e P \hookrightarrow D_\infty \rtimes_{\tau_\infty} G$ by i as well.

Since P is right amenable, G is amenable. Hence there is a canonical faithful conditional expectation \mathcal{E}_∞ from $D_\infty \rtimes_{\tau_\infty} G$ onto D_∞ . Moreover, using Corollary 2.22, we can construct a conditional expectation on $C^{*(\cup)}(P)$ by setting

$$(49) \quad \mathcal{E}^{(\cup)} := (\lambda^{(\cup)}|_{D^{(\cup)}(P)})^{-1} \circ \mathcal{E}_r \circ \lambda^{(\cup)} : C^{*(\cup)}(P) \rightarrow D^{(\cup)}(P).$$

$$\begin{array}{ccc} D^{(\cup)}(P) \rtimes_{\tau^{(\cup)}}^e P & \xrightarrow{i} & D_\infty \rtimes_{\tau_\infty} G \\ \mathcal{E}^{(\cup)} \downarrow & & \downarrow \mathcal{E}_\infty \\ D^{(\cup)}(P) & \xrightarrow{i} & D_\infty \end{array} \quad \text{commutes. But this then}$$

shows that $\mathcal{E}^{(\cup)}$ has to be faithful, and hence that $\lambda^{(\cup)}$ has to be injective (see the Definition of $\mathcal{E}^{(\cup)}$ in (49)). \square

Corollary 4.14. For every cancellative and abelian semigroup P , the canonical homomorphism $\lambda^{(\cup)} : C^{*(\cup)}(P) \rightarrow C_r^*(P)$ is an isomorphism.

Proof. As remarked in [Pa], § (0.18), every abelian semigroup is amenable. \square

As another consequence of Proposition 4.13, we obtain an alternative explanation for the result in [C-D-L] that the Toeplitz algebra over the ring of integers R in some number field can be canonically identified with the reduced semigroup C*-algebra of the $ax + b$ -semigroup P_R over R . First of all, we have proven in Section 2.4 that $\mathfrak{T}[R] \cong C^*(P_R)$. Moreover, we have seen in Lemma 2.30 that the constructible right ideals of P_R are independent, so that $\pi^{(\cup)} : C^*(P_R) \rightarrow C^{*(\cup)}(P_R)$ is an isomorphism. As P_R embeds into the amenable group P_K (the $ax + b$ -group over the quotient field K of R) such that $P_K = P_R^{-1}P_R$, it follows that P_R is cancellative, right reversible (see [Cl-Pr], Theorem 1.24) and hence right amenable (this is the right version of Proposition (1.28) in [Pa]). Therefore, we may apply Proposition 4.13. It tells us that $\lambda^{(\cup)}$ is an isomorphism. All in all, we obtain

$$\mathfrak{T}[R] \cong C^*(P_R) \xrightarrow{\pi^{(\cup)}} C^{*(\cup)}(P) \xrightarrow{\lambda^{(\cup)}} C_r^*(P_R).$$

We point out that the $ax + b$ -semigroup over R is not left reversible.

Moreover, we know from the group case that nuclearity of group C*-algebras is closely related to amenability of groups. Here we show

Proposition 4.15. *Let P be a cancellative, right amenable semigroup. Moreover, assume that P is countable. Then $C^*(P)$, $C^{*(\cup)}(P)$ and $C_r^*(P)$ are nuclear.*

Proof. Since we have surjective homomorphisms $C^*(P) \twoheadrightarrow C^{*(\cup)}(P) \twoheadrightarrow C_r^*(P)$ and because quotients of nuclear C*-algebras are nuclear by [Bla], Corollary IV.3.1.13, it suffices to show that our assumptions imply nuclearity of $C^*(P)$.

Using Lemma 2.14 and dilation theory for semigroup crossed products by endomorphisms (see [La]), we conclude that $C^*(P) \cong D(P) \rtimes_{\tau}^e P \sim_M D_{\infty} \rtimes_{\tau_{\infty}} G$. Here we use analogous notations as in the proof of Proposition 4.13. Now G is amenable as P is right amenable, and D_{∞} is commutative since $D(P)$ is commutative. Hence $D_{\infty} \rtimes_{\tau_{\infty}} G$ is nuclear by [Rør], Proposition 2.12 (i) and (v). Moreover, all the C*-algebras are separable as P is countable. Hence $C^*(P)$ is nuclear because it is stably isomorphic to a nuclear C*-algebra (see [Rør], Proposition 2.12 (ii)). \square

In particular, we obtain because every abelian semigroup is amenable:

Corollary 4.16. *For every countable, cancellative and abelian semigroup P , the C*-algebras $C^*(P)$, $C^{*(\cup)}(P)$ and $C_r^*(P)$ are nuclear.*

In the reverse direction, we can prove

Proposition 4.17. *Let P be a cancellative, left reversible semigroup. If $C_s^*(P)$ or $C^{*(\cup)}(P)$ is nuclear, then P is left amenable.*

Proof. By assumption, P embeds into a group G with $G = PP^{-1}$ (see Theorem 3.10). As P is left reversible, there exists a canonical projection $C_s^*(P) \rightarrow C^*(G)$ sending v_p to u_p . Here u_g , $g \in G$, denote the unitary generators of $C^*(G)$. As nuclearity passes to quotients by [Bla], Corollary IV.3.1.13, nuclearity of $C_s^*(P)$ implies

that $C^*(G)$, hence $C_r^*(G)$ must be nuclear as well. By [Br-Oz], Chapter 2, Theorem 6.8, we conclude that G must be amenable. But a left reversible subsemigroup of an amenable group is itself left amenable by [Pa], (1.28). The analogous proof works also for $C^{*(U)}(P)$ in place of $C_s^*(P)$. \square

5. QUESTIONS AND CONCLUDING REMARKS

An obvious question is: Which semigroups satisfy the condition that their constructible right ideals are independent? It would already be interesting to find out for which integral domains the corresponding $ax + b$ -semigroups satisfy this independence condition.

Another question is whether the condition in Lemma 3.11 is actually necessary. In other words, what is the precise relationship between embeddability of P into a group and the existence of a conditional expectation on $C^*(P)$ satisfying the conclusion in Lemma 3.11?

Furthermore, it would also be interesting to study the question for which subsemigroups of groups the left regular representation $\lambda : C_s^*(P) \rightarrow C_r^*(P)$ is an isomorphism. This is a weaker requirement than left amenability of P . Indeed, we have seen in Section 4 that the difference between the statements “ $\lambda : C_s^*(P) \rightarrow C_r^*(P)$ is an isomorphism” and “ P is left amenable” is precisely given by the property of left reversibility. In this context, A. Nica has studied the example $P = \mathbb{N}^{*n}$, the n -fold free product of \mathbb{N} . He has shown in [Ni], Section 5 that although this semigroup is not left amenable, its left regular representation $\lambda : C^*(\mathbb{N}^{*n}) \rightarrow C_r^*(\mathbb{N}^{*n})$ is an isomorphism. So, the following question remains open: How can we characterize those semigroups which are not left amenable but still satisfy the condition that their left regular representations are isomorphisms?

Finally, let us come back to the construction of semigroup C^* -algebras due to G. Murphy in [Mur2] and [Mur3] mentioned in the introduction. One could say that G. Murphy’s construction leads to very complicated or even not tractable C^* -algebras because the general theory of isometric semigroup representations is extremely complex. If we compare his construction with ours, then we see that G. Murphy’s C^* -algebras encode all isometric representations of the corresponding semigroups whereas representations of our C^* -algebras correspond to rather special isometric representations because of the extra relations we have built into our construction. At the same time, these extra relations lead to a close relationship between our semigroup C^* -algebras and the semigroups themselves in the context of amenability. Such a close relationship does not exist for G. Murphy’s construction. For example, his semigroup C^* -algebra of the semigroup $\mathbb{N} \times \mathbb{N}$ is by definition the universal C^* -algebra generated by two commuting isometries. But this C^* -algebra is not nuclear by [Mur4], Theorem 6.2. Such phenomena cannot occur in our setting by Corollary 4.16.

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